Calculus & D.E.

R.J. Marks II Notes

(1967-1970)

FRESHMAN PLACEMENT EXAMINATION 1967

CARD I

1. The value of $a^2 + 2ab + b^2$, when $a = -5$ and $b = 3$, is:

   a) -32 b) 19 c) -46 d) 4 e) -17

2. a^m divided by a^n =

   a) $a^{m/n}$ b) a^{m-n} c) $(m-n)\log a$ d) $\frac{m}{n} \log a$ e) 1

3. The product of $(5)^2 \cdot (-3)^0 \cdot (-4) \cdot (-2)^3$ =

   a) 800 b) -2400 c) 240 d) 120^6 e) -120^5

4. $(-3a^3)^3$ =

   a) $-3a^6$ b) $3a^9$ c) $-27a^9$ c) $27a^6$ e) $9a^3$

5. Express .0825 as a percent:

   a) 8.25% b) 82 1/2% c) 825% d) .82 1/2% e) .08 1/4%

6. If the dimensions of a rectangle are 3a and 4b, the area is:

   a) 12ab b) 3a + 4b c) $\frac{3a}{4b}$ d) 7ab e) $\frac{a^4b^3}{7}$

7. Simplify:

$$\frac{\frac{1}{a} + \frac{1}{b}}{\frac{1}{a^2} - \frac{1}{b^2}} =$$

   a) $\frac{1}{a} - \frac{1}{b}$ b) $\frac{ab}{b - a}$ c) $a - b$ d) $\frac{a - b}{ab}$ e) 1

8. If $5y^2$ is the <u>quotient</u> and $2xy$ the <u>divisor</u>, the dividend is:

   a) $\frac{5y}{2x}$ b) $y^2 - \frac{2x}{5y}$ c) $5y + 2x$ d) $10xy^3$ e) none of these

9. Simplify $\frac{6a + 10b}{2ab}$:

a) $3 + 5$ b) $\frac{3}{b} + \frac{5}{a}$ c) $12a^2b + 20\ ab^2$ d) $\frac{2ab}{6a + 10b}$

e) none of these

10. Dividing $x^3 - y^3$ by $x - y$ gives

a) $x^2 + y^2$ b) $(x+y)^2$ c) $x^2 + xy + y^2$ d) $(3x-3y)$ e) $(x-y)^2$

11. If $3y$ is an even integer, the next larger consecutive even integer is:

a) $4y$ b) $6y$ c) $3y + 2$ d) $3(y+1)$ e) $y + 3$

12. How many cubic yards of concrete are needed to build a sidewalk x feet long and y feet wide and 4 inches deep?

a) $4xy$ b) $\frac{4xy}{27}$ c) $36xy$ d) $\frac{4}{9}xy$ e) $\frac{1}{81}xy$

13. $3n$ is what percent of $12n$?

a) 4   b) $\frac{1}{4}$ c) 400   d) $9n$ e) 25

14. In $3x^n$, n is called the

a) exponent b) quotient c) logarithm d) dividend e) square

15. If the radius of a circle is doubled, the area is multiplied by:

a) 1   b) $\frac{3}{2}$ c) $\sqrt{2}$ d) 4   e) 2

16. $(a^2+2)^2 =$

a) $a^2 + 4$ b) $a^4 + 4$ c) $a^4 + 4a^2 + 4$ d) $a^4 + 4a + 2$

e) $a^2 + 2a + 4$

17. $\frac{6x + 6}{2x + 2} =$

a) $3 + 3$ b) 3 c) $4x + 4$ d) $3x + \frac{3}{x}$ e) $x + 3$

18. If $3a + b = 5$, $b =$

a) $\frac{5}{3a}$ b) $\frac{3a}{5}$ c) $2 - a$ d) $\frac{5 - a}{3}$ e) $5 - 3a$

19. If $a = -3$, the value of $-2(-3a)^2 =$

a) -162 b) -72 c) -18^2 d) 12^2 e) 18^2

20. Solve for x: $\frac{4x}{5} = \frac{2x + 1}{3} - \frac{4}{15}$

a) $-\frac{3}{2}$ b) $\frac{1}{2}$ c) $\frac{4}{3}$ d) 3 e) none of these

CARD II

21. A house sold for $13,200, which was 25% more than the original cost. The cost was:

a) 11,000 b) 13,000 c) 15,840 d) 9,900 e) 10,560

22. Factor completely: $x^2 - y^2 + 2x + 1$

a) $(x-y)(x+y)(2x+1)$ b) $x(x+2) + (1+y)(1-y)$ c) $(x+1)^2 - y^2$ d) $(x-y)^2(2x+1)$ e) $(x+y+1)(x-y+1)$

23. If the perimeter of a square is p. The area is

a) $(\frac{p}{4})^2$ b) $\frac{p^2}{4}$ c) $\pi\frac{p^2}{4}$ d) p^2 e) πp^2

24. $\frac{x^2+2y}{x^2y+xy^2} - \frac{x-y}{xy} =$

a) $\frac{x^2 - x + y}{x^2y + xy^2 - xy}$ b) $\frac{2}{y + xy} - 1$ c) $\frac{y + 2}{x^2 + xy}$ d) $2y + y^2$ e) $\frac{2y + y^2}{x + y}$

25. $\frac{x - y}{xy} =$

a) 0 b) 1 c) $\frac{1}{y} - \frac{1}{x}$ d) $\frac{xy}{x - y}$ e) none of these

26. $\dfrac{8}{9x^2 - 9x} - \dfrac{5}{6x^2 + 6x} = ?$

a) $\dfrac{3}{3x^2 - 15x}$ b) $\dfrac{13}{3x^2 - 3x}$ c) $\dfrac{3}{15x^2 - 3x}$

d) $\dfrac{3x^2 - 3x}{18x^3 - x}$ e) $\dfrac{x + 31}{18(x^3 - x)}$

27. How many revolutions will a bicycle wheel d feet in diameter make in travelling x feet?

a) $\dfrac{x}{d}$ b) $\dfrac{x}{\pi d}$ c) $\dfrac{\pi x}{d}$ d) $\dfrac{x}{2\pi d}$ e) $\dfrac{2\pi x}{d}$

28. Solve for x: $x^2 - 5x = +6$.

a) 1, -5 b) 5, -6 c) 6, -1 d) $\sqrt{5x + 6}$ e) $\dfrac{+6 + 5x}{x}$

Solve the following inequalities:

29. $7x + 3 > 5x + 6$

a) $x < \frac{3}{2}$ b) $x > 5$ c) $x < -1$ d) $x > \frac{3}{2}$ e) $x < -5$

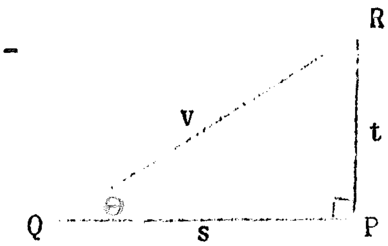
30. $\dfrac{3x + 4}{2} < \dfrac{5x - 6}{2}$

a) $x > \frac{3}{2}$ b) $x < -5$ c) $x < -1$ d) $x < \frac{3}{2}$ e) $x > 5$

31. $\dfrac{2 - 3x}{5} > \dfrac{6 - x}{7}$

a) $x > \frac{3}{2}$ b) $x > 5$ c) $x < -5$ d) $x < \frac{3}{2}$ e) $x < -1$

Given the reference triangle QPR, with the angle PQR denoted by θ, and the angle QPR being a right angle.



32. $\sin\theta =$

a) $\frac{s}{t}$ b) $\frac{v}{s}$ c) $\frac{t}{v}$ d) $\frac{v}{t}$ e) $\frac{s}{v}$

33. $\sec\theta =$

a) $\frac{s}{t}$ b) $\frac{v}{s}$ c) $\frac{t}{v}$ d) $\frac{v}{t}$ e) $\frac{s}{v}$

34. $\cot\theta =$

a) $\frac{s}{t}$ b) $\frac{v}{s}$ c) $\frac{t}{v}$ d) $\frac{v}{t}$ e) $\frac{s}{v}$

35. $\text{arc}\cos\frac{1}{2} =$ (also written $\cos^{-1}(\frac{1}{2})$) $=$?

a) $\frac{\pi}{2}$ b) $\frac{\pi}{3}$ c) $\frac{\pi}{6}$ d) $\frac{\pi}{4}$ e) 0

36. The function which may be expressed $\tan[2\tan^{-1}x] =$ (or $\tan[2\text{ arc }\tan x]$) = ?

a) $\frac{2x}{x^2-1}$ b) $\frac{x^2-1}{2x}$ c) $\frac{2x}{x^2+1}$ d) $\frac{x^2+1}{2x}$ e) $\frac{2x}{1-x^2}$

37. Which of the following is <u>not</u> a "fundamental identity"?

a) $\sin^2\theta + \cos^2\theta = 1$ b) $\frac{\sin\theta}{\cos\theta} = \tan\theta$ c) $\sec\theta = \frac{1}{\cos\theta}$

d) $\cot\theta = \frac{\csc\theta}{\sec\theta}$ e) $\cot^2\theta + 1 = \csc^2\theta$

Solve the following two equations for values of the variable which are positive and not greater than 180^0.

38. $2\sin^2 x - 3\sin x = -1$, $x =$

a) $\frac{\pi}{3}, \frac{\pi}{2}$ b) $\frac{\pi}{3}, 0$ c) $\frac{\pi}{6}, 0$ d) $\frac{\pi}{6}, \frac{\pi}{2}$ e) none of these

39. If $2\sin^2\theta = \cos\theta + 2$, $\theta =$

a) $\frac{\pi}{2}, \frac{2\pi}{3}$ b) $0, \frac{\pi}{3}$ c) $0, \frac{2\pi}{3}$ d) $\frac{\pi}{2}, \frac{\pi}{3}$ e) none of these

40. Evaluate without using tables:

$$\sin(\arctan \tfrac{1}{2} + \arctan \tfrac{1}{3})$$

a) $\frac{1}{\sqrt{2}}$ b) $\frac{3}{\sqrt{5}}$ c) $\frac{2}{\sqrt{10}}$ d) $\frac{13}{\sqrt{50}}$ e) $\frac{2}{3}$

FRESHMAN PLACEMENT EXAMINATION

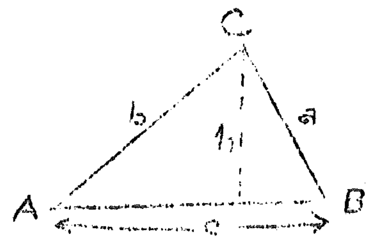
CARD III

1. $\log \frac{a}{b} = ?$

a) $\log a - \log b$ b) $\frac{\log a}{\log b}$ c) $\frac{a}{b}$ d) $\frac{\log a}{b}$

e) $\sqrt[b]{a}$

2. The "law of sines" or "sine law" is:

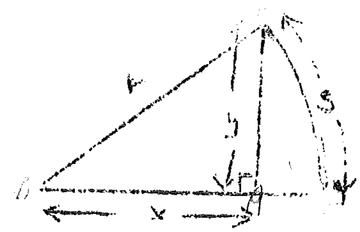


a) $\sin A = \frac{a}{b}$ b) $\sin A = \frac{h}{b}$

c) $\frac{\sin A}{a} = \frac{\sin B}{b}$ d) $\sin A = \frac{h}{a}$

e) $\frac{\sin A}{b} = \frac{\sin B}{a}$

3. In the figure at the right there is an angle θ, a right triangle OAB, and a segment of a circle OCB. The radian measure of the angle θ may be found by dividing:

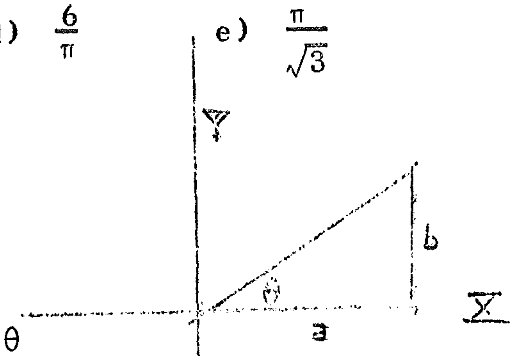


a) y by r b) s by x c) r by s d) x by y e) s by r

4. An angle of 30° has a radian measure of

a) $\frac{\pi}{6}$ b) $\sqrt{3}$ c) $\frac{1}{2}$ d) $\frac{6}{\pi}$ e) $\frac{\pi}{\sqrt{3}}$

5. The complex number $a + ib$ can be expressed in trigonometric form as:



a) $\cos\theta + i \sin\theta$ b) $\sin\theta + i \cos\theta$

c) $r(\cos\theta + i \sin\theta)$ d) $r(\sin\theta + i \cos\theta)$

e) $\cos^2\theta + \sin^2\theta = 1$

6. The three solutions of the equation $x^3 = 8$ are $x = 2$ and the two complex numbers:

a) $-\sqrt{3} \pm i$ b) $-2 \pm 2i\sqrt{3}$ c) $-1 \pm i\sqrt{3}$

d) $-2\sqrt{3} \pm 2i$ e) none of these

7. $\sin(\alpha+\beta) =$

a) $\sin\alpha\cos\beta + \cos\alpha\sin\beta$ b) $\sin\alpha\cos\beta - \cos\alpha\sin\beta$

c) $\cos\alpha\cos\beta + \sin\alpha\sin\beta$ d) $\sin\alpha\sin\beta - \cos\alpha\cos\beta$

e) $\cos\alpha\cos\beta - \sin\alpha\sin\beta$

8. $\cos(\alpha+\beta) =$

a) $\sin\alpha\cos\beta + \cos\alpha\sin\beta$ b) $\sin\alpha\cos\beta - \cos\alpha\sin\beta$

c) $\cos\alpha\cos\beta + \sin\alpha\sin\beta$ d) $\sin\alpha\sin\beta - \cos\alpha\cos\beta$

e) $\cos\alpha\cos\beta - \sin\alpha\sin\beta$

9. $(i^6 + i^9 + i^{12})^5 =$

a) 0 b) i c) 1 d) $-i$ e) -1

10. $(\cos\alpha + i\sin\alpha)^2 =$

a) $\cos 2\alpha + i\sin 2\alpha$ b) $\sin\alpha\cos\alpha - \cos\alpha\sin\alpha$

c) $\cos 2\alpha - i\sin 2\alpha$ d) $\cos^2\alpha - \sin^2\alpha$

e) $\tan^2\alpha$

11. The equation of the circle with radius 1 and center at $(-\frac{3}{2}, \frac{1}{2})$ is

a) $x^2 + y^2 + 3x + y = 1$ b) $2x^2 + 2y^2 + 6x - 2y + 3 = 0$

c) $x^2 + y^2 - 3x - y = 1$ d) $2x^2 + 2y^2 - 6x + 2y + 3 = 0$

e) none of these

12. The slope of the line $\frac{x}{2} - \frac{y}{3} = 1$ is:

a) 1 b) $\frac{2}{3}$ c) $-\frac{3}{2}$ d) $-\frac{2}{3}$ e) $\frac{3}{2}$

13. The distance from (2,2) to the midpoint of the segment joining (2,3) with (-4,-1) is:

a) $2\sqrt{2}$ b) $\sqrt{3}$ c) $3\sqrt{2}$ d) $\sqrt{10}$ e) $\sqrt{13}$

14. The value of k that makes the lines $\begin{cases} 6x - 9y = 5 \\ kx - 4y = 8 \end{cases}$ perpendicular is:

a) -6 b) $-\frac{8}{3}$ c) $\frac{3}{8}$ d) $\frac{8}{3}$ e) 6

15. The parabola whose directrix is the line $y = -1$ and whose focus is $(-1,3)$ is:

a) $x^2 + 2x = 8y$ b) $x^2 + 4x - y = -6$ c) $x^2 = 8y$

d) $2x^2 + 5y = 17$ e) $x^2 + 2x - 8y + 9 = 0$

16. The shortest distance between the circles $x^2 - 6x + y^2 + 5 = 0$, and $x^2 - 8y + y^2 + 15 = 0$ is

a) 0 b) 2 c) 1 d) $\sqrt{5}$ e) 5

In questions 17 and 18 assume the equation of the curve to be

$$4x^2 + 9y^2 + 24x - 18y = 36$$

17. The curve represented is a

a) circle b) parabola c) hyperbola d) ellipse

e) higher plane curve

18. The curve has its center at

a) (12,-9) b) (-12,9) c) (3,-1) d) (-3,1)

e) (6,-2)

In questions 19 and 20, assume the equation of the curve to be

$$4x^2 - 9y^2 + 36 = 0$$

19. The curve represented is a

a) circle b) hyperbola c) parabola d) ellipse

e) higher plane curve

20. As x increases without limit, y becomes

a) zero b) negative c) infinite d) ± 4 e) ± 1

Bob Marks

156

79

147-2) $7X^2 - 4XY + 4Y^2 = 240$

$A = 7 \quad B = -4 \quad C = 4$

$\cot 2\theta = \frac{7-4}{-4} = -\frac{3}{4} \qquad \cos 2\theta = -\frac{3}{5}$

$\sin\theta = \sqrt{\frac{1-\cos 2\theta}{2}} \qquad \cos\theta = \sqrt{\frac{1+\cos 2\theta}{2}}$

$\sin\theta = \sqrt{\frac{4}{5}} \qquad \cos\theta = \sqrt{\frac{1}{5}}$

$X = \frac{1}{\sqrt{5}}X' - \frac{2}{\sqrt{5}}Y' = \frac{X' - 2Y'}{\sqrt{5}}$

$Y = \frac{2}{\sqrt{5}}X' + \frac{1}{\sqrt{5}}Y' = \frac{2X' + Y'}{\sqrt{5}}$

$\frac{7}{5}(X' - 2Y')^2 - \frac{4}{5}(X' - 2Y')(2X' + Y') + \frac{4}{5}(2X' + Y')^2 = 240$

$7(X'^2 - 4X'Y' + 4Y'^2) - 4(2X'^2 - 3X'Y' - 2Y'^2) + 4(4X'^2 + 4X'Y' + Y'^2) = 1200$

$15X'^2 + 40Y'^2 = 1200$

$3X'^2 + 8Y'^2 = 240$

$\frac{X'^2}{80} + \frac{Y'^2}{30} = 1$ ✓

$X'Y'$ coordinates of vertices $(\pm 4\sqrt{5}, 0)$

XY " " " $(4, 8)\ (-4, -8)$ ✓

147-4) $7X^2 - 6XY - Y^2 = 0 \qquad A = 7 \quad B = -6 \quad C = -1$

$\cot 2\theta = -4/3 \qquad \cos 2\theta = -4/5$

$\sin\theta = \sqrt{9/10} \qquad \cos\theta = \sqrt{1/10}$

$X = \frac{X'}{\sqrt{10}} - \frac{3Y'}{\sqrt{10}} \qquad Y = \frac{3X'}{\sqrt{10}} + \frac{Y'}{\sqrt{10}}$

$X = \frac{X' - 3Y'}{\sqrt{10}} \qquad Y = \frac{3X' + Y'}{\sqrt{10}}$

$70(X'^2 - 6X'Y' + 9Y'^2) - 60(3X'^2 - 8X'Y' - 3Y'^2) +$

$-10(9X'^2 + 6X'Y' + Y'^2) = 0$

$70X'^2 - 420X'Y' + 630Y'^2 - 180X'^2 + 480X'Y' + 180Y'^2 - 90X'^2 - 60X'Y' - 10Y'^2 = 0$

$-200X'^2 + 800Y'^2 = 0$

$4Y'^2 = X'^2$

$2Y' = \pm X' \Rightarrow$ 2 straight lines no vertices

in x-y ?

Bob Marks

174-23) PROVE $\log_b X = \dfrac{\log_a X}{\log_a b}$

Let $N = \log_b X \Rightarrow b^N = X$

$\log_a b^N = \log_a X$

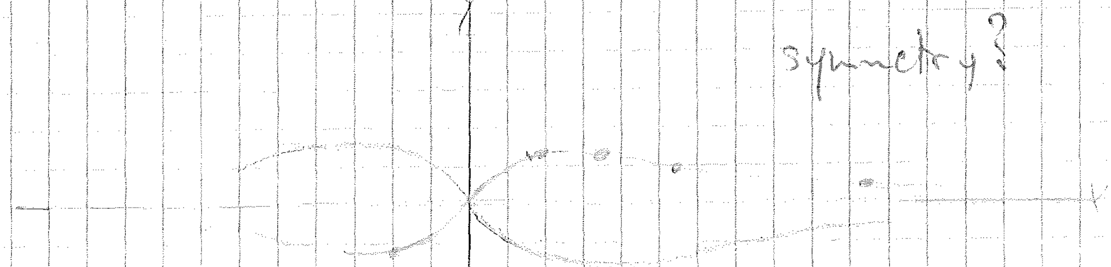
$(N \log_a b = \log_a X)$

$\therefore N = \dfrac{\log_a X}{\log_a b}$

OR $\log_b X = \dfrac{\log_a X}{\log_a b}$

187-26) $r = \cot\theta$

θ	0	$\pi/6$	$\pi/3$	$\pi/2$	$\pi/4$	$\frac{3}{4}\pi$	$\frac{2}{3}\pi$	$\frac{5}{6}\pi$	$\pi/12$	$\pi/18$
θ°	0	30°	60°	90°	45°	135°	120°	150°	15°	10°
$\cot\theta$	—	$\sqrt{3}$	$\sqrt{3}/2$	0	—	-1	$-\sqrt{3}/3$	$\sqrt{3}$	3.7	5.6
r	—	1.730	.58	0	—	-1	$-.58$	-1.734	3.7	5.7



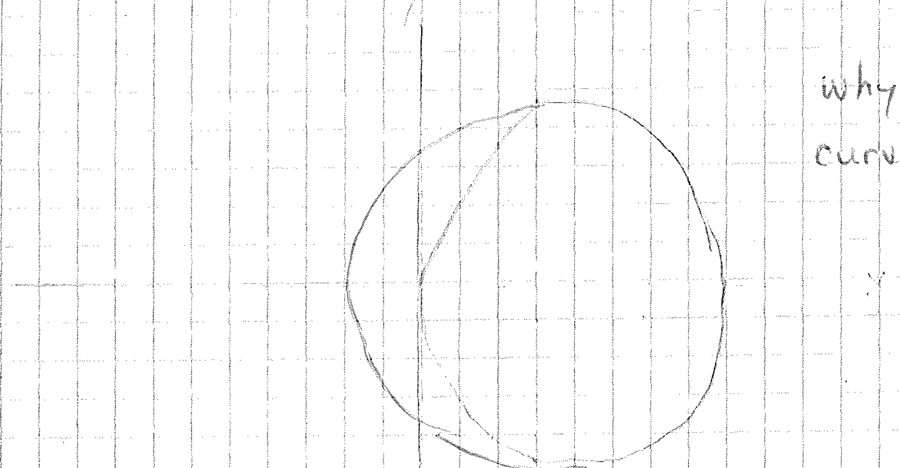
symmetry?

187-28) $r(2-\cos\theta)=4$

symmetry to x axis

$r = \dfrac{4}{2-\cos\theta}$

why draw the curve thru' the pole?



θ	0	$\pi/3$	$\pi/6$	$\pi/4$	$\pi/2$	π	$\frac{3}{4}\pi$	$\frac{2}{3}\pi$	$\frac{5}{6}\pi$	$\frac{15}{?}\pi$
θ°	0°	60°	30°	45°	90°	180°	135°	120°	150°	15°
$\cos\theta$	1	$\frac{1}{2}$	$\sqrt{3}/2$	$\sqrt{2}/2$	0	-1	$-\sqrt{2}/2$	$-\frac{1}{2}$	$-\sqrt{3}/2$	.96
r	4	2.67	3.58	3.14	2	1.33	1.47	1.60	1.55	3.8

Pg 15

1) $|2x+1| = 3$

$2x+1 = {}^{-}3 \qquad 2x+1 = 3$

$2x = {}^{-}4 \qquad 2x = 2$

$x = {}^{-}2 \qquad x = 1$

$x = \{1, {}^{-}2\}$

9) $\left|\frac{2x-3}{3x-2}\right| = 2$

$\frac{2x-3}{3x-2} = 2 \qquad \frac{2x-3}{3x-2} = {}^{-}2$

$6x-4 = 2x-3 \qquad {}^{-}6x+4 = 2x-3$

$4x = 1 \qquad 7 = 8x$

$x = \frac{1}{4} \qquad x = \frac{7}{8}$

$x = \{\frac{1}{4}, \frac{7}{8}\}$

11) $|x-2| < 1$

$U = x-2$

$|U| < 1 \rightarrow U < 1 \cup {}^{-}U < 1$

$x-2 < 1 \cup 2-x < 1$

$x < 3 \cup {}^{-}x < {}^{-}1$

$x < 3 \cup x > 1$

$x = (1, 3)$

18) $\left|\frac{2x-5}{x-6}\right| < 3$

${}^{-}3 < \frac{2x-5}{x-6} < 3 \qquad {}^{-}3 < \frac{2x-5}{x-6} < 3$

$x-6 > 0 \qquad x-6 < 0$

${}^{-}3x+18 < 2x-5 < 3x-18 \qquad {}^{-}3x+18 > 2x-5 > 3x-18$

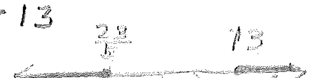
${}^{-}3x+18 < 2x-5 \qquad 2x-5 < 3x-18 \qquad {}^{-}3x+18 > 2x-5 \qquad 2x-5 > 3x-18$

$23 < 5x \qquad 13 < x \qquad 23 > 5x \qquad 13 > x$

$\frac{23}{5} < x \qquad \frac{23}{5} > x$

$x > \frac{23}{5} \qquad x > 13 \qquad x < \frac{23}{5} \qquad x < 13$

$x > 13 \qquad x < \frac{23}{5}$



$(-\infty, \frac{23}{5}) \cup (13, \infty)$

10) $f(x) = \sqrt{(x-1)(x-3)}$

$$(x-1)(x-3) \geq 0$$

$$1 \qquad 3$$

$$(-\infty, 1] \cup [3, \infty)$$

11) $f(x) = \sqrt{2 - 2x - x^2}$

$$2 - 2x - x^2 \geq 0$$

$$x^2 + 2x - 2 \leq 0$$

$$(x^2 + 2x + 1) - 3 \leq 0$$

$$(x+1)^2 \leq 3$$

$$x = \sqrt{3} - 1 \qquad x = -\sqrt{3} - 1$$

$$-\sqrt{3} - 1 \qquad \sqrt{3} - 1$$

$$[-\sqrt{3} - 1, \sqrt{3} - 1]$$

14) $f(x) = x^3$

$$f(x+h) = (x+h)^3$$

$$\frac{(x+h)^3 - x^3}{h} = x^3 +$$

15) $f(x) = \frac{1}{x}$

$$f(x+h) = \frac{1}{x+h}$$

$$\frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$\frac{[x - (x+h)]h}{x^2 + xh} = \frac{-h^2}{x(x+h)}$$

Pg 37

15) $x^2 + xy = 1 \qquad 2x - y = 2 \qquad -y = 2 - 2x \qquad y = 2x - 2$

$$x^2 + x(2x - 2) = 1$$

$$x^2 + 2x^2 - 2x - 1 = 0$$

$$3x^2 - 2x - 1 = 0$$

$$(3x + 1)(x - 1) = 0$$

$$x = -\frac{1}{3} \qquad x = 1$$

$$y = -\frac{8}{3} \qquad y = 0$$

9) $x^2+y^2+4x+6y-21=0$ $\qquad P(-4,5)$

$x_0x+y_0y+2(x_0+x)+3(y_0+y)-21=0$

$-4x+5y+2(x-4)+3(y+5)-21=0$

$-4x+5y+2x-8+3y+15-21=0$

$-2x+8y-14=0$

$-2x=14-8y$

$2x=8y-14$

$x=4y-7$

$(4y-7)^2+y^2+4(4y-7)+6y-21=0$

$16y^2-56y+49+y^2+16y-28+6y-21=0$

$17y^2-46y=0=y(17y-46)$

$y=\{0,\frac{46}{17}\}$

$x=\{-7,\frac{65}{17}\}$

$(-4,5)\ (-7,0)$

$m=\frac{5}{3}$

$y=\frac{5}{3}(x+7)$

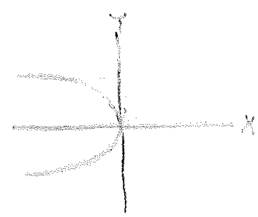
$3y=5x+35$

Pg 108

4) $5y^2=-7x$

$y^2=\frac{-7}{5}x$

$p=\frac{-7}{10}$



focus $=\{0,\frac{-7}{20}\}$

dir $\Rightarrow x=\frac{7}{20}$

Bob Marks

156

79

147-2) $7X^2-4XY+4Y^2=240$

$A=7 \quad B=-4 \quad C=4$

$\cot 2\theta=\frac{7-4}{-4}=-\frac{3}{4}$ $\quad \cos 2\theta=-\frac{3}{5}$

$\sin\theta=\sqrt{\frac{1-\cos 2\theta}{2}}$ $\quad \cos\theta=\sqrt{\frac{1+\cos 2\theta}{2}}$

$\sin\theta=\sqrt{\frac{4}{5}}$ $\quad \cos\theta=\sqrt{\frac{1}{5}}$

$X=\frac{1}{\sqrt{5}}X'-\frac{2}{\sqrt{5}}Y'=\frac{X'-2Y'}{\sqrt{5}}$

$Y=\frac{2}{\sqrt{5}}X'+\frac{1}{\sqrt{5}}Y'=\frac{2X'+Y'}{\sqrt{5}}$

$\frac{7}{5}(X'-2Y')^2-\frac{4}{5}(X'-2Y')(2X'+Y')+\frac{4}{5}(2X'+Y')^2=240$

$7(X'^2-4X'Y'+4Y'^2)-4(2X'^2-3X'Y'-2Y'^2)+4(4X'^2+4X'Y'+Y'^2)=1200$

$15X'^2+40Y'^2=1200$

$3X'^2+8Y'^2=240$

$\frac{X'^2}{80}+\frac{Y'^2}{30}=1$ ✓

$X'Y'$ coordinates of vertices $(\pm 4\sqrt{5}, 0)$

XY " " " $(4,8)\ (-4,-8)$ ✓

147-4) $7X^2-6XY-Y^2=0$ $\quad A=7 \quad B=-6 \quad C=-1$

$\cot 2\theta=-4/3$ $\quad \cos 2\theta=-4/5$

$\sin\theta=\sqrt{9/10}$ $\quad \cos\theta=\sqrt{1/10}$

$X=\frac{X'}{\sqrt{10}}-\frac{3Y'}{\sqrt{10}}$ $\quad Y=\frac{3X'}{\sqrt{10}}+\frac{Y'}{\sqrt{10}}$

$X=\frac{X'-3Y'}{\sqrt{10}}$ $\quad Y=\frac{3X'+Y'}{\sqrt{10}}$

$70(X'^2-6X'Y'+9Y'^2)-60(3X'^2-8X'Y'-3Y'^2)+$
$-10(9X'^2+6X'Y'+Y'^2)=0$

$70X'^2-420X'Y'+630Y'^2-180X'^2+480X'Y'+180Y'^2-90X'^2-60X'Y'-10Y'^2=0$

$-200X'^2+800Y'^2=0$

$4Y'^2=X'^2$

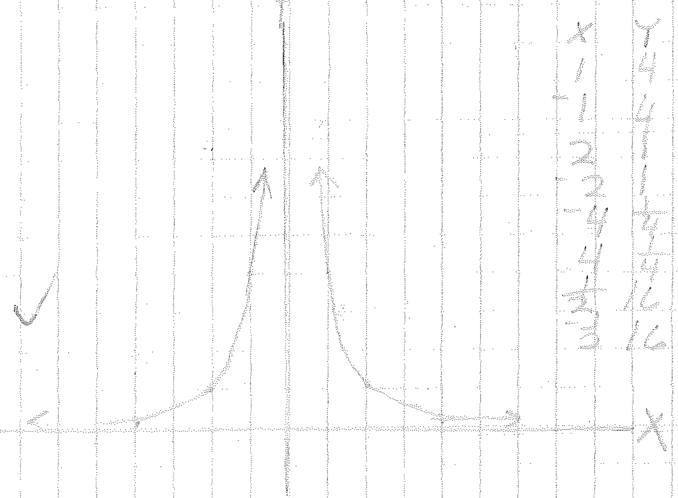
$2Y'=\pm X'$ → 2 straight lines. no vertices

in x-y ?

152-2) $X^2 Y = 4$

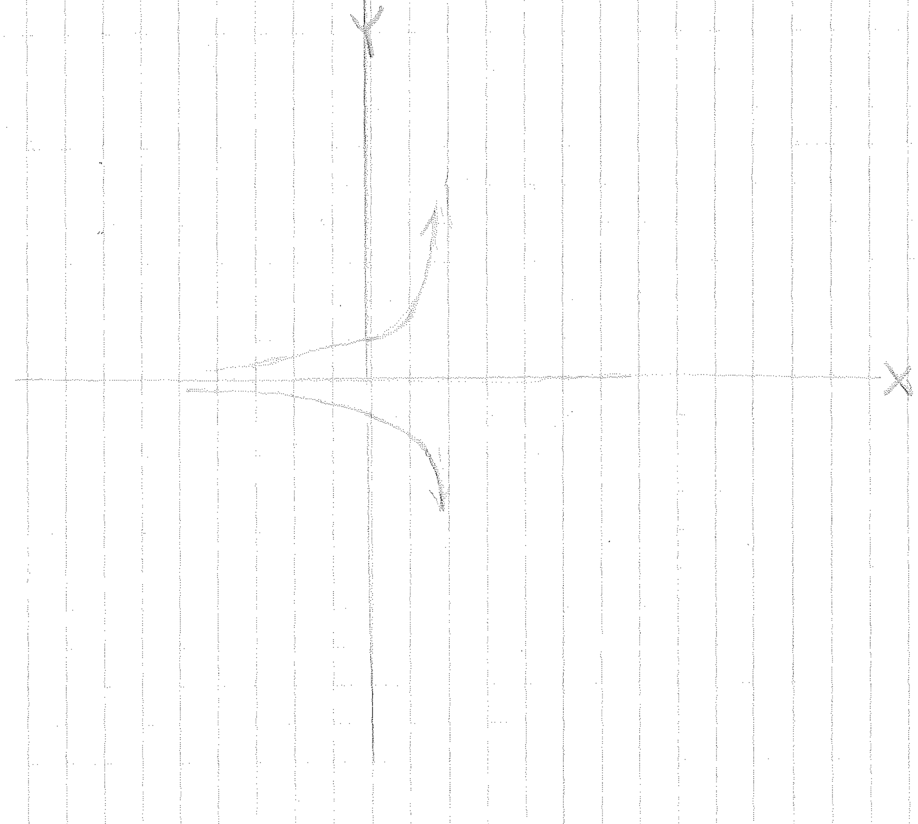
a) symmetry – Y axis
b) domain $Y = 4/X^2 \Rightarrow X \neq 0$
c) range – $X^2 = 4/Y \Rightarrow Y > 0$
d) intercepts $X = 0 \Rightarrow Y = \emptyset \Rightarrow$ NONE
$Y = 0 \Rightarrow X = \emptyset$
e) asymptote – Y axis X axis or $X = 0$ and $Y = 0$

X	Y
1	4
-1	4
2	1
-2	1
4	1/4
1/2	16
-1/2	16



153-14) $Y^2(X-2) + 2 = 0$

a) symm – X axis
b) domain – $Y = \sqrt{\frac{-2}{X-2}} \Rightarrow X < 2$
c) range $X = \frac{2(Y^2-1)}{Y^2} \Rightarrow Y \neq 0$
d) intercepts $Y = 0 \Rightarrow X = \emptyset$
$X = 0 \Rightarrow Y = \pm 1$
e) asymptote X axis $\Rightarrow Y = 0$
and $X = 2$



Bob Marks

174-23) PROVE $\log_b X = \dfrac{\log_a X}{\log_a b}$

Let $N = \log_b X \Rightarrow b^N = X$

$\log_a b^N = \log_a X$

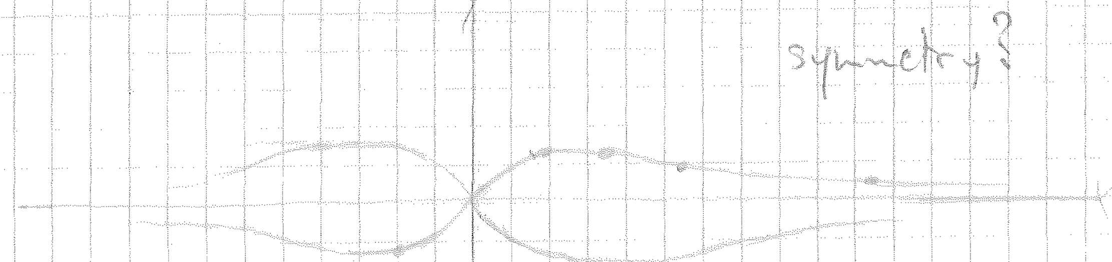
$(N \log_a b = \log_a X)$

$\therefore N = \dfrac{\log_a X}{\log_a b}$

OR $\log_b X = \dfrac{\log_a X}{\log_a b}$

187-26) $r = \cot\theta$

θ	0	$\pi/6$	$\pi/3$	$\pi/2$	$\pi/4$	$\frac{3}{4}\pi$	$\frac{2}{3}\pi$	$\frac{5}{6}\pi$	$\pi/12$	$\pi/18$
θ°	0	30°	60°	90°	45°	135°	120°	150°	15°	10°
$\cot\theta$	—	$\sqrt{3}$	$\sqrt{3}/3$	0	—	-1	$-\sqrt{3}/3$	$-\sqrt{3}$	3.7	5.7
r	—	1.739	$+.58$	0	—	-1	$-.58$	-1.734	3.7	5.7



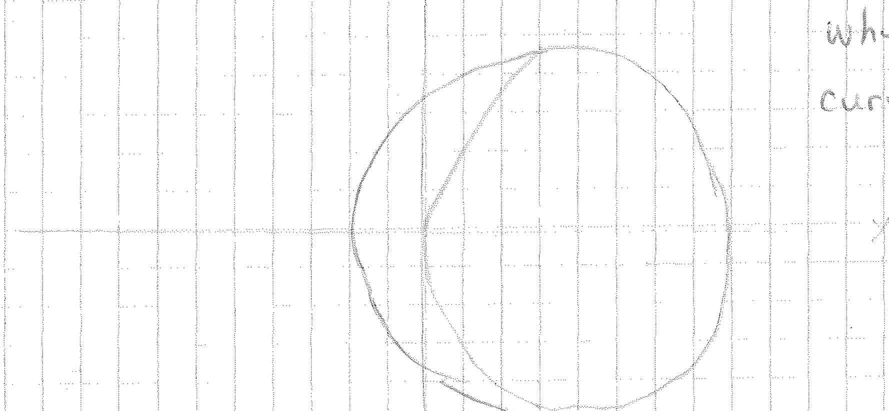
symmetry?

187-28) $r(2 - \cos\theta) = 4$

symmetry to x axis

$r = \dfrac{4}{2 - \cos\theta}$

why draw the curve thru' the pole?



θ	0	$\pi/3$	$\pi/6$	$\pi/4$	$\pi/2$	π	$\frac{3}{4}\pi$	$\frac{2\pi}{3}$	$\frac{5}{6}\pi$	$\frac{\pi}{12}$
θ°	0°	60°	30°	45°	90°	180°	135°	120°	150°	15
$\cos\theta$	1	$\frac{1}{2}$	$\sqrt{3}/2$	$\sqrt{2}/2$	0	-1	$-\sqrt{2}/2$	$-\frac{1}{2}$	$-\sqrt{3}/2$	.
r	4	2.67	3.53	3.14	2	1.33	1.48	1.60	1.55	3.

189-22) $r^2 \sin 2\theta = 2$

$r^2 = 2/\sin 2\theta$

$= 2/2\sin\theta\cos\theta$

$r^2 \sin^2\theta = \sin\theta/\cos\theta$

$y^2 = \frac{\sin\theta}{\cos\theta} = \tan\theta = \frac{y}{x} \Rightarrow$

$y^2 = y/x$

$y^2 x = y$

$xy = 1$

189-28) $r = a\cos 3\theta$

$r^2 = a^2\cos^2 3\theta$

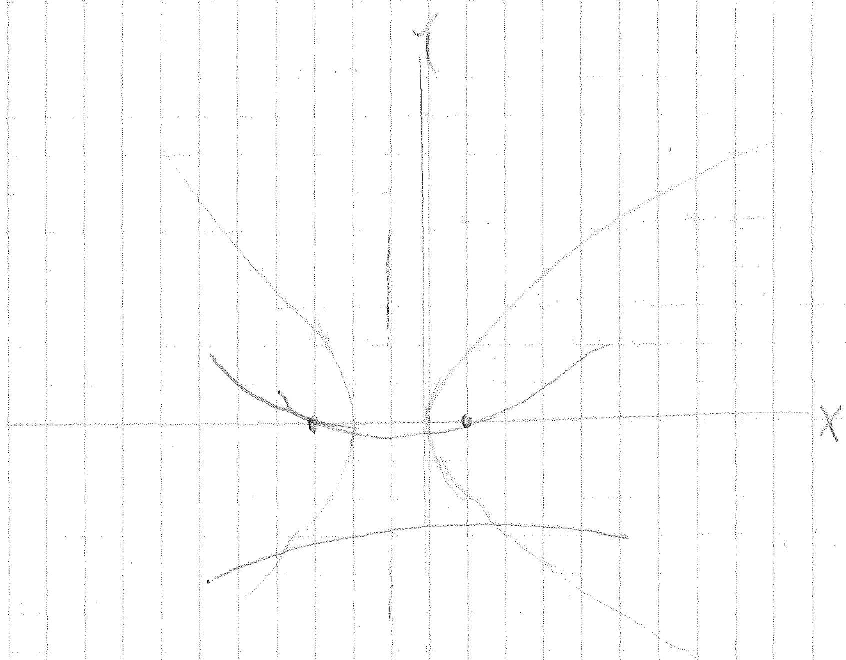
$x^2+y^2 = a^2\cos^2 3(\arctan \frac{y}{x})$

192-28) $r = \frac{4}{2-4\sin\theta}$ $e = 2$ $p = 1$

$r \sin\theta = -1$

or directrix $\Rightarrow y = -1$

hyperbola



PROF.
HOFSOMMER

Pg 15

1) $|2x+1| = 3$

$2x + 1 = {}^-3 \qquad 2x+1=3$

$2x = {}^-4 \qquad 2x = 2$

$x = {}^-2 \qquad x = 1$

$x = \{1, {}^-2\}$

9) $\left|\frac{2x-3}{3x-2}\right| = 2$

$\frac{2x-3}{3x-2} = 2 \qquad \frac{2x-3}{3x-2} = {}^-2$

$6x - 4 = 2x - 3 \qquad {}^-6x + 4 = 2x - 3$

$4x = 1 \qquad 7 = 8x$

$x = \frac{1}{4} \qquad x = \frac{7}{8}$

$x = \{\frac{1}{4}, \frac{7}{8}\}$

11) $|x-2| < 1$

$0 = x-2$

$|0| < 1 \rightarrow 0 < 1 \cup {}^-0 < 1$

$x - 2 < 1 \quad \cup \quad 2 - x < 1$

$x < 3 \cup -x < {}^-1$

$x < 3 \cup x > 1$

$x = (1, 3)$

18) $\left|\frac{2x-5}{x-6}\right| < 3$

${}^-3 < \frac{2x-5}{x-6} < 3 \qquad {}^-3 < \frac{2x-5}{x-6} < 3$

$x-6 > 0 \qquad x - 6 < 0$

${}^-3x + 18 < 2x - 5 < 3x - 18 \qquad {}^-3x + 18 > 2x - 5 > 3x - 18$

${}^-3x + 18 < 2x - 5 \quad 2x - 5 < 3x - 18 \quad {}^-3x + 18 > 2x - 5 \quad 2x - 5 > 3x - 18$

$23 < 5x \quad 13 < x \quad 23 > 5x \quad 13 > x$

$\frac{23}{5} < x \qquad \frac{23}{5} > x$

$x > \frac{23}{5} \quad x > 13 \quad x < \frac{23}{5} \quad x < 13$

$x > 13 \qquad x < \frac{23}{5}$

$(-\infty, \frac{23}{5}) \cup (13, \infty)$

19) $\left|\frac{x+3}{6-5x}\right| \leq 2$

$-2 \leq \frac{x+3}{6-5x} \leq 2$ $\qquad$ $\cancel{-2} \leq \frac{x+3}{6-5x} \leq 2$

$6-5x>0$ $\qquad$ $6-5x<0$

$-2 \leq \frac{x+3}{6-5x}$ $\qquad$ $\frac{x+3}{6-5x} \leq 2$

$-12+10x \leq x+3$ $\qquad$ $x+3 \leq 12-10x$

$9x \leq 15$ $\qquad$ $11x \leq 9$

$x \leq \frac{5}{3}$ $\qquad$ $x \leq \frac{9}{11}$

$x \leq \frac{9}{11}$ $\qquad$ $x \geq \frac{5}{3}$

$(-\infty, \frac{9}{11}] \cup [\frac{5}{3}, \infty)$

21) $|x+3| \leq |2x-6|$

$|x+3| = |2x-6|$

$|x+3| = 2x-6$ $\qquad$ $|x+3| = 6-2x$

$x+3 = 2x-6$ $\qquad$ $\cancel{-x-3 = 2x-6}$ $\qquad$ $x+3 = 6-2x$ $\qquad$ $\cancel{-x-3 = 6-2x}$

$9 = x$ $\qquad$ $3x = 3 \Rightarrow x = 1$

$(-\infty, 1] \cup [9, \infty)$

Pg 29

$\cancel{11)}$ $Y = \sqrt{2-2x-x^2}$

$2-2x-x^2 \geq 0$

$x^2+2x-2 \leq 0$

$Y = (2-2x-x^2)^{\frac{1}{2}}$

$U = 2-2x-x^2$ $\qquad$ $Y = U^{\frac{1}{2}}$

$\frac{dx}{dx} = (-2-2x)\frac{1}{2}($

$x^2+2x-2=0$

$Y = \sqrt{2-2x-x^2}$ $= \sqrt{-1x^2-2x+2}$

$= \sqrt{-1(x^2+2x-2)}$

$= \sqrt{-1[(x+1)^2-3]}$

$(x+1)^2 = 3$

$x+1 = \sqrt{3}$

$x = \sqrt{3}-1$

$y > \sqrt{3}-1$

10) $f(x) = \sqrt{(x-1)(x-3)}$

$(x-1)(x-3) \geq 0$

$1 \qquad 3$

$(-\infty, 1] \cup [3, \infty)$

11) $f(x) = \sqrt{2 - 2x - x^2}$

$2 - 2x - x^2 \geq 0$

$x^2 + 2x - 2 \leq 0$

$(x^2 + 2x + 1) - 3 \leq 0$

$(x+1)^2 \leq 3$

$x = \sqrt{3} - 1 \qquad x = -\sqrt{3} - 1$

$-\sqrt{3} - 1 \qquad \sqrt{3} - 1$

$[-\sqrt{3} - 1, \sqrt{3} - 1]$

14) $f(x) = x^3$

$f(x+h) = (x+h)^3$

$\dfrac{(x+h)^3 - x^3}{h} = x^3 +$

15) $f(x) = \dfrac{1}{x}$

$f(x+h) = \dfrac{1}{x+h}$

$\dfrac{\frac{1}{x+h} - \frac{1}{x}}{h}$

$\dfrac{[x-(x+h)]h}{x^2+xh} = \dfrac{-h^2}{x(x+h)}$

Pg 37

15) $x^2 + xy = 1 \qquad 2x - y = 2 \qquad -y = 2 - 2x \qquad y = 2x - 2$

$x^2 + x(2x-2) = 1$

$x^2 + 2x^2 - 2x - 1 = 0$

$3x^2 - 2x - 1 = 0$

$(3x+1)(x-1) = 0$

$x = -\frac{1}{3} \qquad x = 1$

$y = -\frac{8}{3} \qquad y = 0$

Pg 55

25) $X + 5Y - 3 = 0$

$5Y = -X + 3$

$Y = -\frac{1}{5}X + \frac{3}{5}$

$m = -\frac{1}{5}$

$Y + 4 = -\frac{1}{5}(X - 1)$

$Y + 4 = -\frac{1}{5}X + \frac{1}{5}$

$5Y + 20 = -X + 1$

$5Y + X + 19 = 0$

Pg 65

7) $Y + 2 = 0$

$Y = -2$

$m = 0$

$d = 2$

Pg 67

3) $(2,0)$

$Y = m(X - 2)$

7) $Y = -X$

Pg 101

1) $X^2 + Y^2 + 2X - 4Y = 0$ $P(1,3)$

$X_0X + Y_0Y + (X + X_0) - 2Y - 2Y_0 = 0$

$X + 3Y + X + 1 - 2Y - 6 = 0$

$2X + Y - 5 = 0$

7) $X^2 + Y^2 - 4X - 7Y - 6 = 0$ $P(6,1)$

$X_0X + Y_0Y - 2(X + X_0) - \frac{7}{2}(Y_0 + Y) - 6 = 0$

$6X + Y - 2X - 12 - \frac{7}{2}Y - 21 - 6 = 0$

$-\frac{5}{2}Y - 4X - 39 = 0$

$-5Y = 8X + 78$

$m = -\frac{8}{5}$

$Y - 1 = \frac{5}{8}(X - 6)$

$Y - 1 = \frac{5}{8}X - \frac{30}{8}$

$8Y - 8 = 5X - 30$

$8Y + 5X - 38 = 0$

9) $x^2+y^2+4x+6y-21=0$ $\qquad P(-4,5)$

$x_0x+y_0y+2(x_0+x)+3(y_0+y)-21=0$

$-4x+5y+2(x-4)+3(y+5)-21=0$

$-4x+5y+2x-8+3y+15-21=0$

$-2x+8y-14=0$

$-2x=14-8y$

$2x=8y-14$

$x=4y-7$

$(4y-7)^2+y^2+4(4y-7)+6y-21=0$

$16y^2-56y+49+y^2+16y-28+6y-21=0$

$17y^2-46y=0=y(17y-46)$

$y=\{0, \frac{46}{17}\}$

$x=\{-7, \frac{65}{17}\}$

$(-4,5)\ (-7,0)$

$m=\frac{5}{3}$

$y=\frac{5}{3}(x+7)$

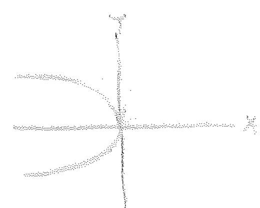
$3y=5x+35$

Pg 108

5) $5y^2=-7x$

$y^2=-\frac{7}{5}x$

$p=-\frac{7}{10}$

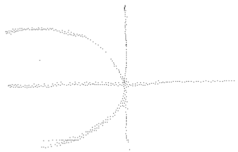


focus $=\{0, -\frac{7}{20}\}$

dir $\Rightarrow x=\frac{7}{20}$

3) $Y^2 = -12X$

$P = -6$



focus $= \{-3, 0\}$

dir $\Rightarrow X = 3$

1) $Y^2 = 8X$

$P = 4$

focus $= \{2, 0\}$

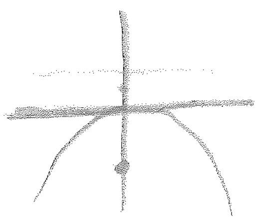
dir $\Rightarrow X = -2$

7)

$P = -6$

$Y^2 = -12X$

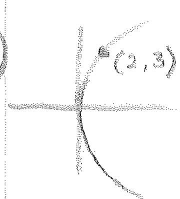
9)



$P = 4$

$X^2 = -8Y$

13)



$Y^2 = 2PX$

$9 = 4P$

$P = \frac{9}{4}$

$Y^2 = \frac{9}{2}X$

17) $Y = -2X^2 + 4X + 5$

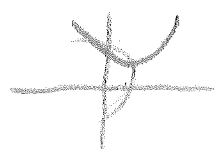
$Y = -2(X^2 - 4X - \frac{5}{2})$

$-\frac{1}{2}Y = [(X^2 - 4X + 4) + \frac{3}{2}]$

$(X-2)^2 = -\frac{1}{2}Y + \frac{3}{2}$

$(X-2)^2 = -\frac{1}{2}(Y-3)$

vertex $= \{2, 3\}$



Robert Marks

50 A/B

Pg 68

16) $\begin{array}{l} 3X + 4Y - 2 = 0 \\ -(2X + Y - 6 = 0) \end{array} \Rightarrow \begin{array}{l} 6X + 8Y - 4 = 0 \\ -(6X + 3Y - 18) = 0 \end{array}$

$5Y + 14 = 0$

$5Y = -14$

$Y = -14/5$

$2X - \frac{14}{5} - 6 = 0$

$2X = \frac{44}{5}$

$X = 22/5$

$\left\{ \frac{22}{5}, \frac{-14}{5} \right\}$

$Y - Y_1 = m(X - X_1)$

$Y + \frac{14}{5} = -2\left(X - \frac{22}{5}\right)$

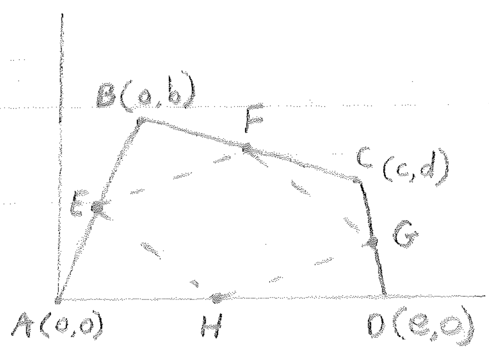
$Y + \frac{14}{5} = -2X + \frac{44}{5}$

$Y = -2X + \frac{30}{5}$

$Y = -2X + 6$

Pg 73

4)



Of E, F, G, H are midpoints =

$E = \left(\frac{a}{2}, \frac{b}{2}\right)$

$F = \left(\frac{a+c}{2}, \frac{b+d}{2}\right)$

$G = \left(\frac{c+e}{2}, \frac{d}{2}\right)$

$H = \left(\frac{e}{2}, 0\right)$

$m = \frac{Y_2 - Y_1}{X_2 - X_1}$

$m \text{ of } \overline{EF} = \frac{\frac{b}{2} - \left(\frac{b+d}{2}\right)}{\frac{a}{2} - \left(\frac{a+c}{2}\right)} = \frac{d}{c}$

$m \text{ of } \overline{HG} = \frac{\frac{d}{2} - 0}{\frac{c+e}{2} - \frac{e}{2}} = \frac{d}{c}$

$m_{\overline{EF}} = m_{\overline{HG}} \Rightarrow \overline{EF} \parallel \overline{HG}$ ($\parallel$ = IS PARALLEL TO)

$m \text{ of } \overline{FG} = \frac{\frac{b+d}{2} - \frac{d}{2}}{\frac{a+c}{2} - \frac{c+e}{2}} = \frac{b}{a-e}$

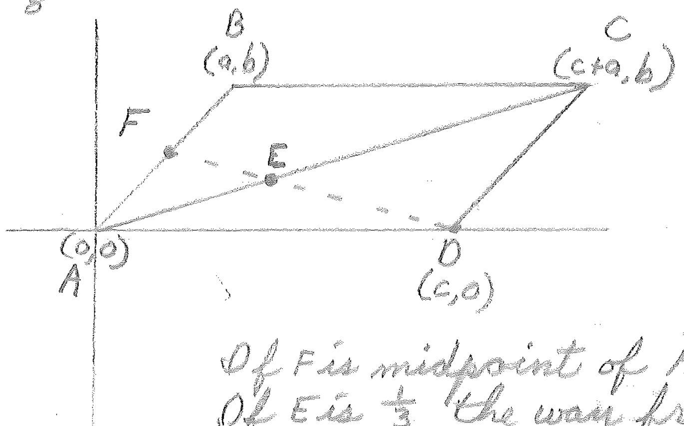
$m \text{ of } \overline{EH} = \frac{\frac{b}{2}}{\frac{a}{2} - \frac{e}{2}} = \frac{b}{a-e}$

$m_{\overline{FG}} = m_{\overline{EH}} \Rightarrow FG \parallel EH$

$\overline{FG} \parallel \overline{EH} \cup \overline{EF} \parallel \overline{HG} \Rightarrow$ EFGH IS A PARALLELOGRAM

Pg 74

12)



Of F is midpoint of $\overline{AB}$, $F = \{\frac{a}{2}, \frac{b}{2}\}$
Of E is $\frac{1}{3}$ the way from A to C, $E = \{\frac{c+a}{3}, \frac{b}{3}\}$

Of $\overline{FED}$ is a straight line, $m_{\overline{DE}} = m_{\overline{DF}}$ because m of a line is constant

$m = \frac{Y_2 - Y_1}{X_2 - X_1}$

$m_{\overline{DE}} = \frac{\frac{b}{3}}{\frac{c+a}{3} - c} = \frac{b}{a-2c}$

$m_{\overline{DF}} = \frac{\frac{b}{2}}{\frac{a}{2} - c} = \frac{b}{a-2c}$

$\therefore m_{\overline{DE}} = m_{\overline{DF}} \Leftrightarrow$ D,E,F lie on the same st. line

Pg 97

30) $X^2 + Y^2 + DX + EY + F = 0$

a) $\{2,0\}$: $2D + 0 + F = {}^-4$
b) $\{0,4\}$: $0 + 4E + F = {}^-16$
c) $\{2,2\}$: $2D + 2E + F = {}^-8$

a) − c) $\Rightarrow {}^-2E = 4$
$E = {}^-2$
b) $\Rightarrow 4({}^-2) + F = {}^-16$
$F = {}^-8$
a) $\Rightarrow 2D - 8 = {}^-4$
$D = 2$
d) $\{1,1\}$: $D + E + F = {}^-2$
Of $\{1,1\}$ IS ON CIRCLE, $(2) + ({}^-2) + ({}^-8) = {}^-2$
but ${}^-8 \neq {}^-2$
$\therefore$ $\{2,0\}$, $\{0,4\}$, $\{2,2\}$, and $\{1,1\}$ do not lie on a circle

BOB MARKS

x) $2X - 3Y + 6 = 0$

$3Y = 2X + 6$

$Y = \frac{2}{3}X + 2 \quad \therefore m = \frac{2}{3}$

$Y - Y_1 = m(X - X_1)$ where $(X_1, Y_1) = \{3, 1\}$ and $m = \frac{2}{3}$

$Y - 1 = \frac{2}{3}(X - 3)$

$Y - 1 = \frac{2}{3}X - 2$

$Y = \frac{2}{3}X - 1$

$3Y = 2X - 3$ ✓

If lines $a \perp b$ then $m_a(m_b) = {}^-1$

$Y - 1 = \frac{^-3}{2}(X - 3)$

$Y - 1 = \frac{^-3}{2}X + \frac{9}{2}$

$2Y - 2 = {}^-3X + 9$

$2Y = {}^-3X + 11$ ✓

15) $Y = X^2 - 2X + 3$

$X^2 - 2X = Y - 3$

$X^2 - 2X + 1 = Y - 2$

$(X-1)^2 = (Y-2)$

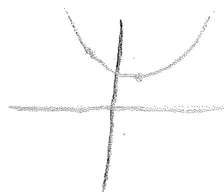
vertex $= \{1, 2\}$

$p = \frac{1}{2}$

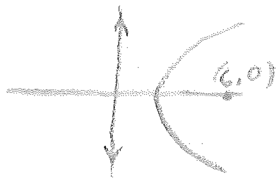
axis $\Rightarrow X = 1$

focus $= \{1, 2\frac{1}{4}\}$

direc $\Rightarrow Y = 1\frac{3}{4}$



23)



vertex $= \{3, 0\}$

$Y^2 = 12(X-3)$

$Y^2 = 12X - 36$

Pg 113

1) $Y^2 = 9X \quad P(1,3)$

$Y_0 Y = \frac{9}{2}(X + X_0)$

$3Y = \frac{9}{2}(X+1)$

$6Y = 9X + 9$

$2Y = 3X + 3$

11) $Y^2 = -4X \quad P(1,5)$

$5Y = -2X - 2$

$Y = \frac{-1}{5}(2X+2)$

$\left[-\frac{1}{5}(2X+2)\right]^2 = -4X$

$\frac{1}{25}(4X^2 + 8X + 4) = -4X$

$Y^2 = -4\left(\frac{-5}{2}Y - 1\right)$

$= 10Y + 4$

$Y^2 - 10Y + 4 = 0$

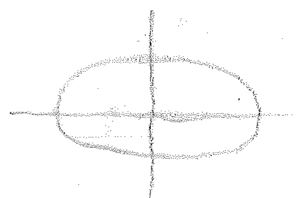
$Y = \frac{10 \pm \sqrt{100 - 16}}{2} = \frac{10 \pm 2\sqrt{21}}{2}$

11) $Y^2 = {}^-4x$ $\qquad Y-5 = m(x-1)$ $\qquad P(1,5)$

$$Y - 5 = mx - m$$
$$\frac{Y-5+m}{m} = X$$
$$mY^2 = {}^-4Y + 20 - 4m$$
$$mY^2 + 4Y + 4m - 20 = 0$$
$$16 - 4m(4m-20) = 0$$
$$16 - 16m^2 + 80m = 0$$
$$-m^2 + 5m + 1 = 0$$
$$m^2 - 5m - 1 = 0$$
$$m = \frac{5 \pm \sqrt{25+4}}{2}$$
$$m = \frac{5 \pm \sqrt{29}}{2}$$
$$2Y - 10 = (5+\sqrt{29})(x-1)$$
$$2Y - 10 = (5+\sqrt{29})X - 5 - \sqrt{29}$$
$$2Y = (5+\sqrt{29})X + 5 - \sqrt{29} \text{ etc}$$

Pg 118

9) $2x^2 + 3Y^2 = 11$



$$\frac{x^2}{3} + \frac{Y^2}{2} = \frac{11}{6}$$
$$\frac{x^2}{\frac{33}{6}} + \frac{Y^2}{\frac{22}{6}} = 1$$
$$\frac{x^2}{\frac{11}{2}} + \frac{Y^2}{\frac{11}{3}} = 1$$

$a = \sqrt{\frac{11}{2}}$ $\quad b = \sqrt{\frac{11}{3}}$

$$\frac{11}{2} = \frac{11}{3} + c^2$$
$$c^2 = \frac{11}{2} - \frac{11}{3} = \frac{33}{6} - \frac{22}{6} = \frac{11}{6}$$
$$c = \sqrt{\frac{11}{6}}$$

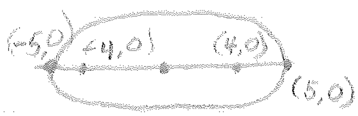
foci $= \left\{\pm\sqrt{\frac{11}{6}}, 0\right\}$

vert $= \left\{\pm\sqrt{\frac{11}{2}}, 0\right\}$

maj. axis $= 2\sqrt{\frac{11}{2}}$

min. axis $= 2\sqrt{\frac{11}{3}}$

$$e = \frac{c}{a} = \frac{\sqrt{\frac{11}{6}}}{\sqrt{\frac{11}{2}}} = \frac{\sqrt{\frac{11}{6}}}{\sqrt{\frac{33}{6}}} = \frac{\sqrt{11}}{\sqrt{33}} = \frac{1}{\sqrt{3}}$$

11) 

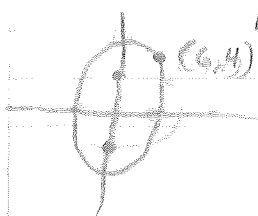
$a=5 \quad c=4 \quad b=3$

$$\frac{x^2}{25}+\frac{y^2}{9}=1$$

$$9x^2+25y^2=225$$

21) $e=\frac{c}{a}=\frac{3}{4}$ $\quad \frac{b}{a}=\frac{\sqrt{7}}{4}$ $\quad \frac{y^2}{a^2}+\frac{x^2}{b^2}=1$ $\quad P(6,4)$

$b=\frac{\sqrt{7}a}{4}$ $\quad \frac{16}{a^2}+\frac{36}{b^2}=1$



$$16b^2+36a^2=a^2b^2$$

$$16\left(\frac{7a^2}{16}\right)+36a^2=a^2\left(\frac{7a^2}{16}\right)$$

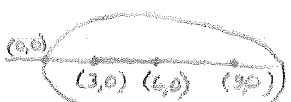
$$7a^2+36a^2=\frac{7a^4}{16}$$

$$7a^2+36a^2-\frac{7a^4}{16}=0$$

$$7+36=\frac{7a^2}{16}$$

$$\frac{43(16)}{7}=a^2$$

$$4\sqrt{\frac{43}{7}}=a$$

23) 

center $=(6,0)$

$c=3 \quad a=6 \quad b^2=27$

$\cancel{\frac{x^2}{36}+\frac{y^2}{27}=1}$ $\qquad \frac{(x-6)^2}{4}+\frac{y^2}{3}=1$

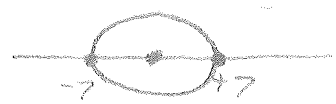
$\cancel{\frac{x^2}{4}+\frac{y^2}{3}=1}$ $\qquad 3(x-6)^2+4y^2=108$

$\cancel{3x^2+4y^2=12}$ $\qquad \cancel{3(x^2-12x+36)+4y^2-12=0}$

$\cancel{3x^2-36x+108+4y^2-12=0}$

$3x^2+4y^2-36x+96=0$

Pg 123

5) 

$a = 7$

$\frac{c}{a} = e$

$dir = \frac{a}{e} = \frac{a^2}{c} = 12 = \frac{49}{c}$

$12c = 49$

$c = \frac{49}{12}$

$49 - \left(\frac{49}{12}\right)^2 = b^2$

$\frac{7056 - 2041}{(12)^2} = \frac{5015}{(12)^2}$

$b = \frac{\sqrt{5015}}{12}$

$\frac{x^2}{49} + \frac{y^2 144}{5015} = 1$

11) $\frac{x^2}{9} + \frac{y^2}{4} = 1$

$4x^2 + 9y^2 = 36$

$4(x) + 9y\left(-\frac{4}{3}\sqrt{2}\right) = 36$

$4x - 12\sqrt{2}\,y - 36 = 0$

$x - 3\sqrt{2}\,y - 9 = 0$

21) $\frac{x^2}{4} + \frac{y^2}{2} = 1 \quad P(1,4) \quad P(x_0\, y_0)$

$2x^2 + 4y^2 = 8$

$x^2 + 2y^2 = 4$

$xx_0 + 2yy_0 = 4$

$x_0 + 8y_0 = 4$

$x_0 = 4 - 8y_0$

$2x_0^2 + 4y_0^2 = 8$

$\cancel{2}(4 - 8y_0)^2 + \cancel{4}2y_0^2 = \cancel{8}4$

$\cancel{2}(16 - 32y_0 + 64y_0^2 + 2y_0^2 = 4$

$66y_0^2 - 32y_0 + 12 = 0$

$33y_0^2 - 16y_0 + 6 = 0$ ~~etc~~

$(\cancel{3x_0 - 3})(11y_0 - 2$

Pg 161

1) $Y = 2 \sin \frac{1}{2} X$

X	0	30°	45°	60°	90°	120°	135°	150°	180°	270°	720°
X	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3}{4}\pi$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
Y	0			1	1.4	1.7			2	1.4	0

$Y = 2 \sin \frac{1}{2}\left(\frac{\pi}{3}\right) = 2 \sin \frac{\pi}{6} = 1$

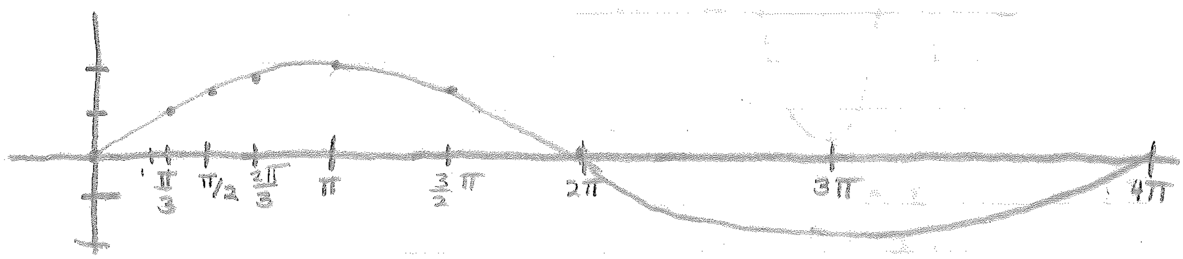
$Y = 2 \sin \frac{1}{2}\left(\frac{\pi}{2}\right) = 2 \sin \frac{\pi}{4} = \frac{2}{\sqrt{2}} \cong 1.4$

$Y = 2 \sin \frac{1}{2}\left(\frac{2\pi}{3}\right) = 2 \sin \frac{\pi}{3} = \frac{2\sqrt{3}}{2} = 1.7$

$Y = 2 \sin \frac{1}{2}(\pi) = 2 \sin \frac{\pi}{2} = 2$

$Y = 2 \sin \frac{1}{2}\left(\frac{3}{2}\pi\right) = 2 \sin \frac{3\pi}{4} = \frac{2}{\sqrt{2}} = 1.4$

$Y = 2 \sin \frac{1}{2}(2\pi) = 2 \sin \pi = 0$



Period 4π
Amplitude = 2

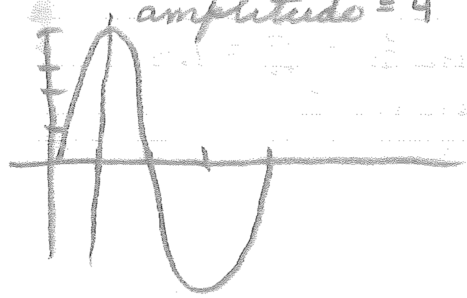
3) $4 \cos \frac{1}{2} \pi X = f(x) \cong 4 \cos \frac{1}{2} (3.14)$

X	0	30°	45°	60°	90°	120°	135°	150°	180°
X	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
Y									

3) $4 \cos \frac{1}{2} \pi X$

$\left(\frac{2\pi}{\frac{1}{2}\pi}\right)$ = period = 4

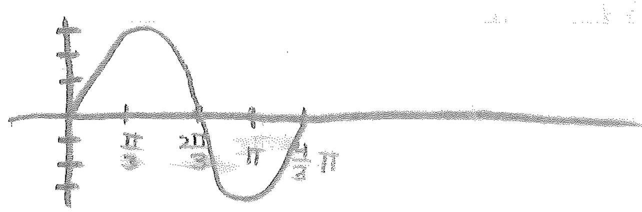
amplitude = 4



5) $3 \sin \frac{3}{2} X$

amp = 3

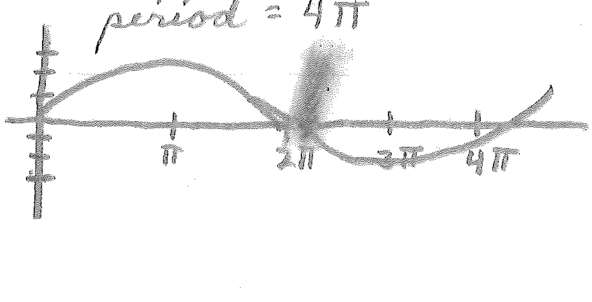
period = $\frac{2\pi}{\frac{3}{2}} = \frac{4}{3}\pi$



13) $-3 \sin \left(\frac{1}{2}X + 7\right)$

amplitude = 3

period = 4π



Pg 186

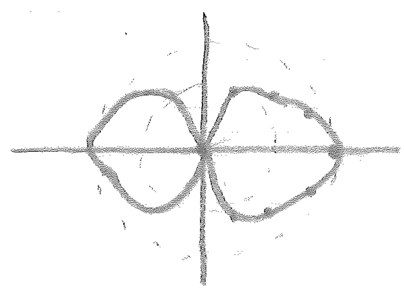
1) $r = 2\cos\theta$

$2\cos\theta = 2\cos{^-\theta}$ -sym, with x axis

$2\cos\theta \neq 2\cos(\pi - \theta)$

$2\cos\theta \neq 2\cos(\pi + \theta)$

θ	0	30°	45°	60°	90°	270°	300°	315°	330°	360°
$r =$	2	$\sqrt{3}$	$\sqrt{2}$	1	0	0	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	$\sqrt{3}$	2
r	2	1.7	1.4	1	0	0	1.2	1.4	1.7	2



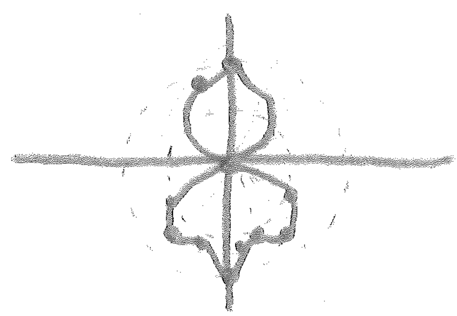
3) $r = {^-2}\sin\theta$

${^-2}\sin\theta \neq {^-2}\sin{^-\theta}$

${^-2}\sin\theta = {^-2}\sin(\pi - \theta)$ sym with Y axis

${^-2}\sin\theta \neq {^-2}\sin(\theta + \pi)$

θ	90°	0°	30°	45°	60°	270°	300°	315°
r	-2	0	-1	$\frac{-2}{\sqrt{2}}$	$\frac{-2}{\sqrt{3}}$	-2	$-\sqrt{3}$	
r	-2	0	-1	-1.4	-1.2	-2	-1.7	



Tangents to a Quadratic Curve

Rule Let $Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0$ be the equation of a quadratic curve and let $P(p,q)$ be a point on the curve. Then the tangent at P can be found from the equation of the curve by carrying out the following substitutions:

$x^2 \longrightarrow px$ $\qquad\qquad$ $x \longrightarrow \frac{1}{2}(x+p)$

$xy \longrightarrow \frac{1}{2}(qx + py)$ $\qquad\qquad$ $y \longrightarrow \frac{1}{2}(y+q)$

$y^2 \longrightarrow qy$

Caution The above rule has been proven and only holds for the quadratic curve $Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0$!!

Example Find the equation of the lines tangent to $y^2 - 3y + 4x - 8 = 0$ at the points where $x = 1$.

Solution The point lies on the curve. Its y-coordinate is obtained by substitution of $x = 1$ in its equation and solving for y: $y^2 - 3y - 4 = (y-4)(y+1) \Rightarrow y = 4$ or $y = -1$. The two points $P(1,4)$ and $Q(1,-1)$ are on the curve.

For $P(1,4)$ substitute $y^2 \to 4y$, $x \to \frac{1}{2}(x+1)$, $y \to \frac{1}{2}(y+4)$.

Hence $4y - \frac{3}{2}y - 6 + 2x + 2 - 8 = 2x + \frac{5}{2}y - 12 = 0$ is the tangent at P.

For $Q(1,-1)$ substitute $y^2 \to -y$, $x \to \frac{1}{2}(x+1)$, $y \to \frac{1}{2}(y-1)$

Hence $-y - \frac{3}{2}y + \frac{3}{2} + 2x + 2 - 8 = 2x - \frac{5}{2}y - \frac{9}{2} = 0$ is the tangent at Q.

Example Find the lines through $P(-5,2)$ tangent to $y^2 - y - 2x + 4 = 0$.

Solution Let $Q(p,q)$ be the point of contact. The tangent at Q has equation $2qy - q - y - 2p - 2x + 8 = 0$ (1). Since $P(-5,2)$ is on the tangent: $3q - 2p + 16 = 0$ or $2p = 3q + 16$, (2) which gives us one relation between p and q. Since Q is on the curve we have $q^2 - q - 2p + 4 = 0$, (3) giving another relation between p and q. Solving p and q from (2) and (3) yields the two points $Q_1(-5,-2)$ and $Q_2(17,6)$. Substitution of the obtained values of p and q into (1) finally yields the two tangents $2x + 5y = 0$ and $2x - 11y + 32 = 0$.

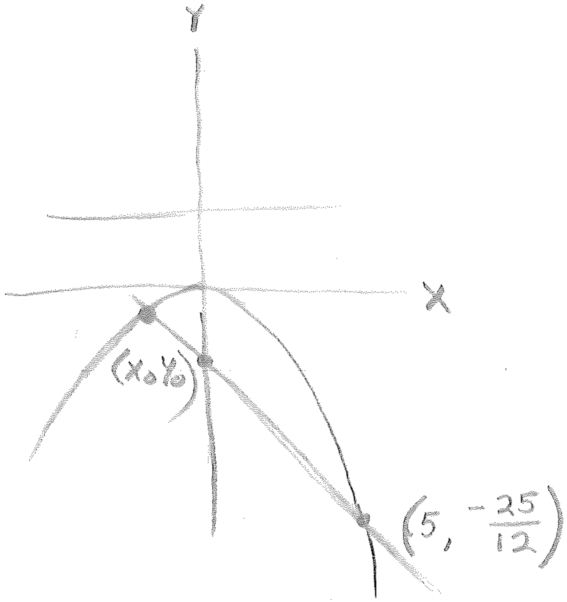
40

C^+

Test 19 Nov 1968 Analytic Geometry Box No. 156

Name BOB MARKS

1. A line through the focus of the parabola $x^2 = -12y$ intersects the parabola at the point $(5, -25/12)$. Find the other point of intersection of this line with the parabola.



-2

(OVER)

$X^2 = 2PY$

$P = 6$

$\therefore f = \{0, -3\}$ ✓ $\{5, \frac{-25}{12})$

$Y - Y_1 = m(X - X_1)$

$m = \frac{Y_2 - Y_1}{X_2 - X_1} = \frac{\frac{-25}{12} + 3}{-5}$

$= \frac{\frac{-25}{12} + \frac{36}{12}}{+5}$

$= \frac{36 - 25}{60} = \frac{11}{60}$

$Y + 3 = \frac{11}{60}(X)$

$60Y + 180 = 11X$

$X^2 = -12Y$

$Y = \frac{X^2}{-12}$

$-60\left(\frac{X^2}{12}\right) + 180 = 11X$

$-5X^2 - 11X + 180 = 0$

$X = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

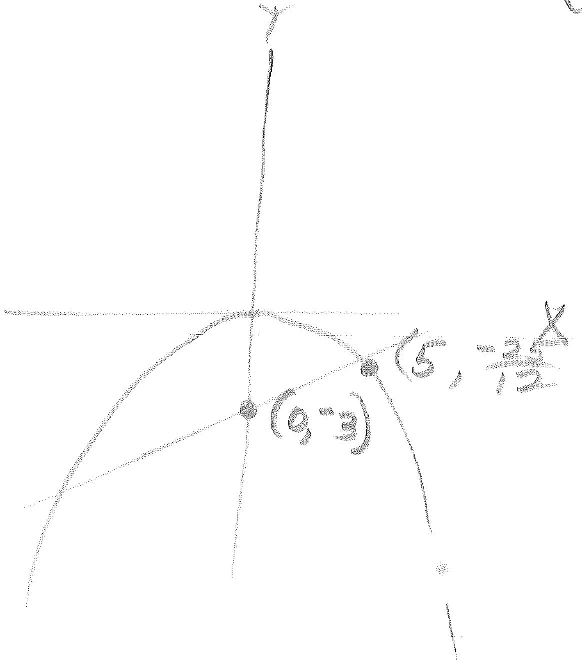
$= \frac{11 \pm 61}{-10}$

$X = \frac{72}{-10}$ ✓

$60Y + 180 = \frac{-792}{10}$

$60Y$

$x^2 = -12Y$



$\left(5, \frac{-25}{12}\right)$

$x^2 = 2PY$

$P = 6$

$foci = (0, -3)$

$Y - Y_1 = m(x - X_1)$

$m = \frac{Y_2 - Y_1}{X_2 - X_1} = \frac{-3 + \frac{25}{12}}{-5}$

$m = \frac{11}{60}$

$Y + 3 = \frac{11}{60}(X)$

$60Y + 180 = 11X \qquad x^2 = -12Y$

$Y = \frac{-x^2}{12}$

$\frac{-60x^2}{12} - 11X + 180 = 0$

$-5x^2 - 11X + 180 = 0$

$5x^2 + 11X - 180 = 0$

3721

$X = \frac{-11 \pm \sqrt{121 + 3600}}{10}$

$= \frac{-11 \pm 61}{10}$

$X = \frac{-72}{10} = \frac{-36}{5}$

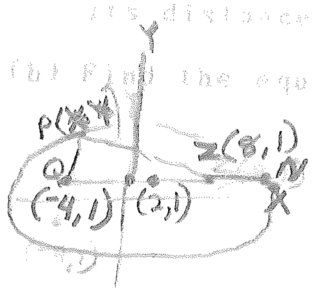
$Y + 3 = \frac{-11}{5}\left(\frac{3}{5}\right)$

$Y = \frac{-11}{25} - \frac{75}{25} = \frac{-86}{25}$

$\left\{\frac{-36}{5}, \frac{-86}{25}\right\}$

2. (a) Find the locus of a point which moves so that the sum of its distances from (8, 1) and (-4, 1) is always 20.

(b) Find the equations of the directrices of this conic.



ellipse

$2a = 12 = (\overline{PZ})$ $\quad c = 6$

$\overline{QZ} + 2\overline{ZN} = 20 \Rightarrow a = 10$

$\overline{ZN} = 4 \Rightarrow a = 10$

$b^2 + c^2 = a^2$

$b^2 = 100 - 36$

$b^2 = 64$

$C_{\text{ENTER}} = \{2, 1\}$

-2

$$\frac{(X-2)^2}{100} + \frac{(Y-1)^2}{64} = 1$$

$64(X-2)^2 + 100(Y-1)^2 = 6400$

$16(X-2)^2 + 25(Y-1)^2 = 1600$

$16(X^2 - 4X + 4) + 25(Y^2 - 2Y + 1) = 1600$

$\cancel{16X^2 - 64X} + 64 + \cancel{25Y^2 - 50Y} + 25 = 1600$

$16X^2 - 64X + 25Y^2 - 50Y = 1511$

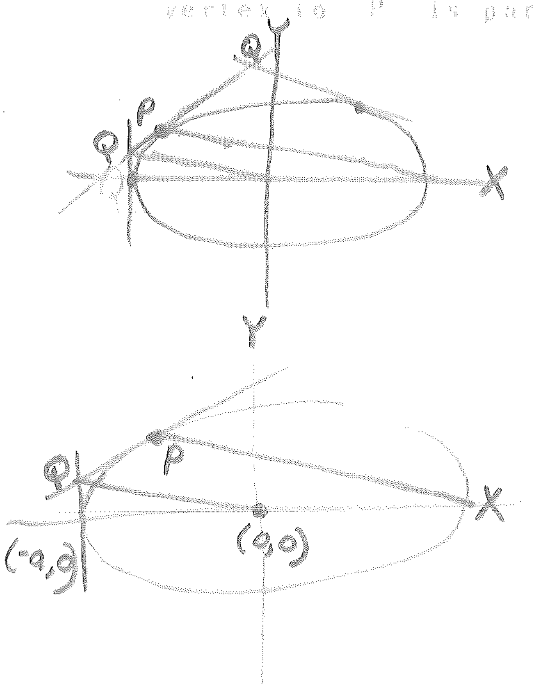
DIRECT

$|d| = \frac{a}{e} = \frac{a^2}{c} = \frac{100}{6} = \frac{50}{3}$

$X = \frac{50}{3}$

$X = \frac{-44}{3}$

1. The tangent to an ellipse at a point P meets the tangent at a vertex in a point Q. Prove that the line joining the other vertex to P is parallel to the line joining the center to Q.



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

-17

4. Given the hyperbola $(x^2/4) - y^2 = 1$.

(a) Find the equation of the tangent line at the point $(5/2, 3/4)$.

(b) Find the equations of the lines tangent to this hyperbola passing through the point $(1, 2)$.

a) $\frac{x^2}{4} - y^2 = 1$

$\frac{x x_0}{4} - y y_0 = 1 \qquad (x_0, y_0) = \left(\frac{5}{2}, \frac{3}{4}\right)$

$\frac{\frac{5}{2}x}{4} - \frac{3}{4}y = 1$

$\frac{5x}{8} - \frac{3y}{4} = 1$

$5x - 6y = 8$

b) $\frac{x^2}{4} - y^2 = 1 \qquad y - 2 = m(x - 1)$

$y = mx - m + 2$

$\frac{x^2}{4} - (mx - m$ [illegible] $)^2 = 1$

-9

$\frac{x^2}{4} - mx^2$

A N A L Y T I C   G E O M E T R Y

F I N A L   E X A M I N A T I O N

B

Rose Polytechnic Institute
9 December 1968

NAME Bob Marks Box No 156

INSTRUCTOR Hoffmann Section C

INSTRUCTIONS

1. Work ALL EIGHT (8) problems of Part I (i.e., problems 1 through 8).
2. Work ANY TWO (2) problems of Part II (i.e., problems A through G).
3. Cross out (><) any problem which is not to be graded even if you did not try it. If you work more than two problems of Part II and fail to cross out those to be left ungraded, then the first two you worked (in alphabetical order) will be the only ones graded.

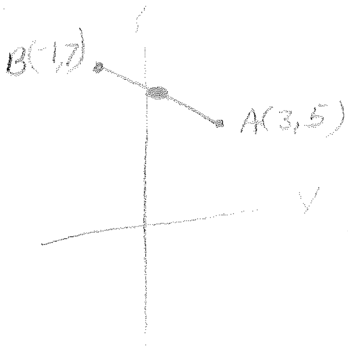
= =

(Please Do Not Write Below This Line)

PART I	1	10	(10)	PART II	A	10	(10)
	2	10	(10)		B	—	(10)
	3a	4	(5)		Ca	4	(4)
	b	5	(5)		b	1	(2)
	4	10	(10)		c	4	(4)
	5a	4	(5)		Da	—	(5)
	b	4			b	—	(5)
	6	7	(10)		E	—	(10)
	7a	4	(4)		F	—	(10)
	b	0	(3)		G	—	(10)
	c	1	(3)				
	8a	3	(5)				
	b	5	(5)				

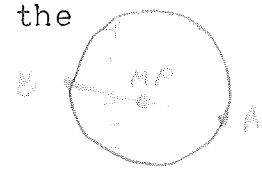
TOTAL 86 (100)

1



Name Bob Marks Box No 156

1. Find the equation of the circle which has as a diameter the line segment AB, where A is (3,5) and B is (-1,7).



Center = $(1,6) = (h,k)$

$r^2 = (3-1)^2 + (5-6)^2 = 4+1 = 5$

$(x-h)^2 + (y-k)^2 = r^2$

$(x-1)^2 + (y-6)^2 = 5$

$\cancel{x^2-2x}+1+y^2-12y+36=5$

$x^2-2x+y^2-12y+32=0$ ✓

2. Find the point on the line x = 2 which is equidistant from the points (3,4) and (-2,7).

$(-2,7)$ $(3,4)$ MP of PQ = $\left(\frac{1}{2}, \frac{11}{2}\right)$

m. of PQ

$m = \frac{4-7}{3+2} = -\frac{3}{5}$

$\perp$ to PQ

$\left(y - \frac{11}{2}\right) = \frac{5}{3}\left(x - \frac{1}{2}\right)$

$3y - \frac{33}{2} = 5x - \frac{5}{2}$

$6y - 33 = 10x - 5 \Rightarrow 6y - 10x - 28 = 0$

$x = 2$

$\therefore 6y - 20 - 28 = 0$

$6y = 48$

$y = \frac{48}{6} = 8$

$P(2,8)$ ✓

Name ___Marks___ Box No 156

3. Given the three lines $2x - 3y + 7 = 0$, $3x + 2y + 4 = 0$, and $x - 8y + 36 = 0$:

a. Prove that these three lines form a right triangle.

$3X + 2Y + 4 = 0$ $\quad 2Y = -3X - 4$ $\quad Y = -\frac{3}{2}X - 2$ $\quad m_1 = -\frac{3}{2}$

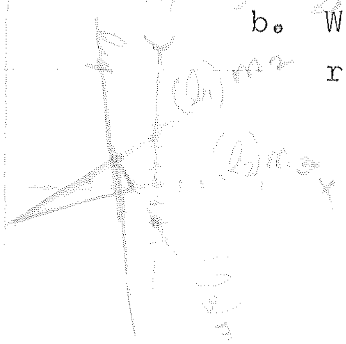
$2X - 3Y + 7 = 0$ $\quad 3Y = 2X + 7$ $\quad Y = \frac{2}{3}X + \frac{7}{3}$ $\quad m_2 = \frac{2}{3}$ ✓

$X - 8Y + 36 = 0$ $\quad 8Y = X + 36$ $\quad Y = \frac{1}{8}X + \frac{9}{2}$ $\quad m_3 = \frac{1}{8}$

but is there a triangle?

$m_1 \neq m_2 \neq m_3$ they aren't ... $m_2 m_1 = -1$ ∴ the 2 lines are ⊥

b. Write the equation of the line through the vertex of the right angle and perpendicular to the hypotenuse.



$3X + 2Y + 4 = 0$ $\quad 2X - 3Y + 7 = 0$

$6X + 4Y + 8 = 0$
$6X - 9Y + 21 = 0$
$13Y = 13$
$Y = 1$
$3X + 2 + 4 = 0$
$3X = -6$
$X = -2$

pt of intersection of l_1 & $l_2 = \{-2, 1\}$ ✓

m of hypotenuse $= m_3 = \frac{1}{8}$

∴ m of $l_4 = -8$

$Y - 1 = -8(X + 2)$
$Y - 1 = -8X - 16$
$Y + 8X + 15 = 0$ ✓

4. Find the equation of the line parallel to the line $3x + y - 17 = 0$ which passes through the intersection of the two lines $3x - 5y + 6 = 0$ and $2x + y - 9 = 0$.

l_1) $3X + Y - 17 = 0$
$Y = 17 - 3X$
$m_1 = -3$

l_2) $3X - 5Y + 6 = 0$
$5Y = 3X + 6$

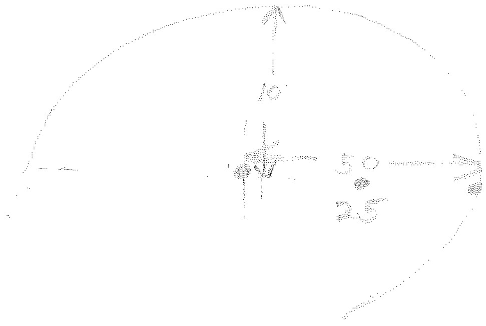
l_3) $2X + Y - 9 = 0$
$Y = 9 - 2X$
$5Y = 45 - 10X$

$3X + 6 = 45 - 10X$
$13X = 39$
$X = 3$
$6 + Y - 9 = 0$
$Y = 3$

int. of l_2 and $l_3 = P = \{3, 3\}$

$Y - 3 = -3(X - 3)$
$Y - 3 = -3X + 9$
$Y + 3X - 12 = 0$ ✓

6) a)



$a = 50 \qquad b = \cancel{10}\ 20$

$$\frac{X^2}{2500} + \frac{Y^2}{100} = 1$$

$$100X^2 + 2500Y^2 = 250000$$

$$X^2 + 25Y^2 = 2500$$

(assigning origin at mid. [illegible] [illegible] [illegible])

b) $X = 25$

$$(25)^2 + 25(Y^2) = 2500$$

$$25 + Y^2 = 100$$

$$Y^2 = 75$$

$$Y = \sqrt{75}$$

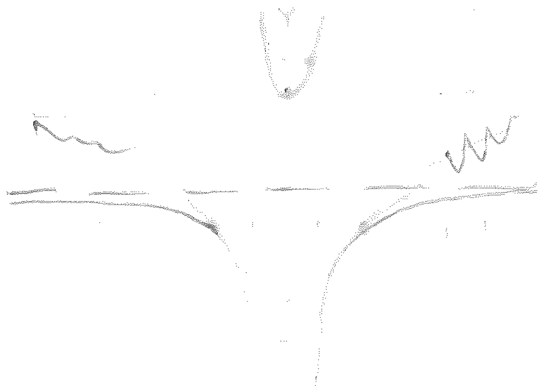
$$= 5\sqrt{3}\ ft.$$

Name Bob [illegible] Box NO 156

5. Discuss and sketch each of the following:

a. $y = \dfrac{x^2-4}{x^2-1} = \dfrac{(x-2)(x+2)}{(x-1)(x+1)}$

[illegible] at $x = \pm 1$ and $y = 1$
intercepts $(2,0)$ $(-2,0)$ $(0,4)$
domain = all $x \neq \pm 1$
range ~~= all y~~
symmetric with Y axis



b. $y = \sqrt{\dfrac{2+x^2}{1-x^2}}$

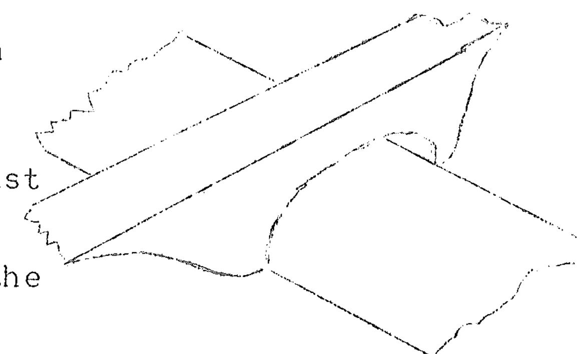
As
Range $y > \emptyset$ $\sqrt{2}$
Domain $(-1,1)$ ✓
Asymptotes $x = \pm 1$ ✓
Y inter. $(0, \sqrt{2})$
no X inter.
sym. with Y axis ✓

$\dfrac{2+x^2}{1-x^2} \geq 0$

$-1 < x < 1$

$x = \pm \frac{1}{2}$
$y = 3$

6. An overpass is to be built on an elliptic arch having a span of 100 ft and a clearance of 20 ft at the center (see sketch). In order to pour the concrete to cast this arch, it is necessary to build the forms by knowing the height at various points along the span. Write an equation which could be used for this purpose and determine the height at a point 25 ft from one end.



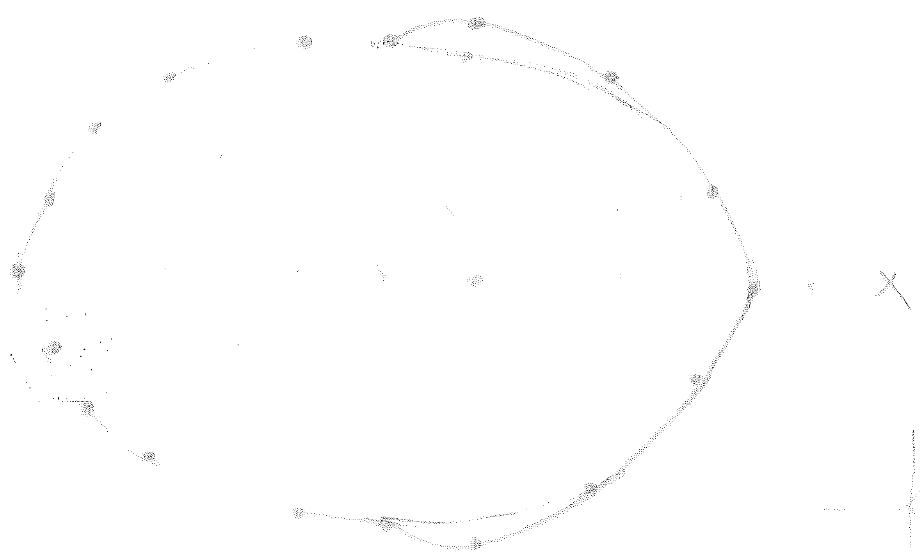
8)a) $r = 4 + \cos 2\theta$

θ	$0°$	$30°$	$45°$	$60°$	$90°$	$150°$	$210°$	$180°$	$195°$	225
r	5	$4\frac{1}{2}$	4	$3\frac{1}{2}$	3	$-\frac{\sqrt{3}}{2}+4$	$4\frac{1}{2}$	5	$\frac{\sqrt{3}}{2}+4$	4
r	5	4.5	4	3.5	3	~~3.2~~ 3.5	4.5	5	~~4.8~~ 4.5	4

$\cos 2\theta = \cos 2(-\theta)$ – symmetric with X axis

$\cos 2\theta \overset{?}{=} \cos 2(\pi - \theta)$

$\cos 2\theta \overset{?}{=} \cos 2(\pi + \theta)$



Name ______________ Box No 156

7. Determine the solution set (i.e., solve for x) for each of the following inequalities:

a. $x^2 + 2x - 15 > 0$

$(x+5)(x-3)>0$

$-5 \qquad 3$

$(-\infty,-5)\cup(3,\infty)$ ✓

b. $|x-7| < 2|x-10|$

for $x>10$: $x-7 < 2x-20$, $13 < x$, $x>13$

$7-x < 2x-10$, $17 < 3x$, $x > \frac{17}{3}$

for $7<x<10$: $x-7 < 20-2x$

for $x<7$: $7-x < 20-2x$

$3 \qquad \frac{17}{3}$

$(-\infty, \frac{17}{3})$

c. $x+2 < |2x+3|$

for $x > -\frac{3}{2}$: $x+2 < 2x+3$, $-1 < x$

$-x-2 < 2x+3$, $-5 < 3x$, $x > -\frac{5}{3}$

$x+2 < -2x-3$ for $x < -\frac{3}{2}$

$3x < -5$

$x < -\frac{5}{3}$

$-\frac{5}{3} \qquad -1$

$(-\infty, -\frac{5}{3}) \cup (-1, \infty)$

correct by mistake.

8. a. Discuss and sketch the polar curve $r = 4 + \cos 2\theta$.

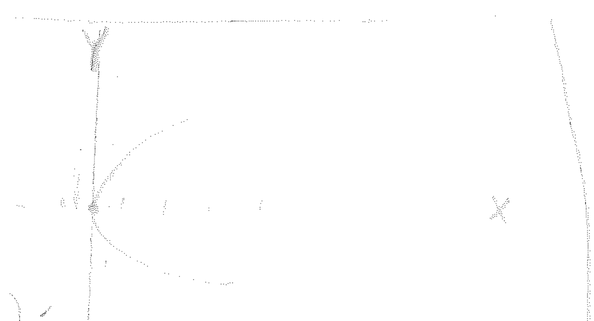
$r = \frac{2\cos\theta}{\sin^2\theta}$ $= 2\cot$ / $\sin\theta$ $\quad \cot = \frac{x}{y}$

$r\sin\theta = 2\frac{x}{y}$

$y = 2\frac{x}{y}$

$y^2 = 2x$ ✓

a parabola $\quad$ focus $= (\frac{1}{2}, 0)$ ✓ $\quad$ directrix $(x = -\frac{1}{2})$ ✓



b. Transform the polar equation $r = 2\,\frac{\cos\theta}{\sin^2\theta}$ into rectangular coordinates, identify the curve, and sketch it.

$y = r\sin\theta$

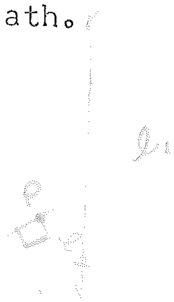
$x = r\cos\theta$

$r = y$

Name Bob Marks Box No 156

PART II [Cross out all problems of this part which are not to be graded.]

A. A point, whose path in the xy-plane is a straight line, comes closest to the origin at $P(-\frac{3}{2},2)$. Write the equation of the path.



m of $l_2 = \dfrac{2}{-\frac{3}{2}} = -\left(\frac{4}{3}\right)$

$\therefore m$ of $l_1 = \frac{3}{4}$

$Y - 2 = \frac{3}{4}\left(X + \frac{3}{2}\right)$

$Y - 2 = \frac{3}{4}X + \frac{9}{8}$

$8Y - 16 = 6X + 9$

$8Y - 6X - 25 = 0$ ✓

~~B. Find the equation(s) of the line(s) through the point (-3,-5) making with the line $3x - y - 5 = 0$ an angle whose tangent is 2. [Hint: $\tan(\theta - \phi) = \dfrac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi}$.]~~

c) a) $Y^2 - 4X - 4Y + 16 = 0$

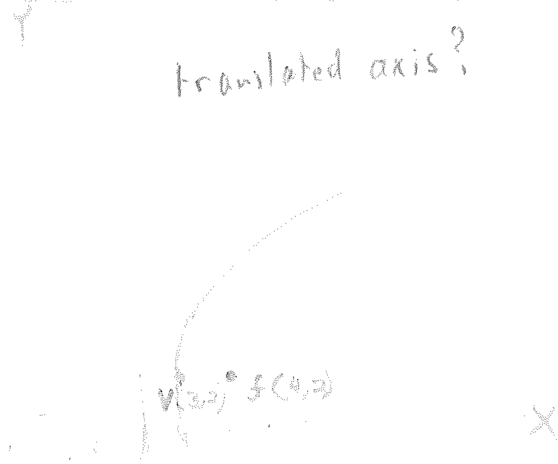
$Y^2 - 4Y = 4X - 16$

$Y^2 - 4Y + 4 = 4X - 12$

$(Y-2)^2 = 4(X-3)$

$P = 2$

translated axis?



axis $\Rightarrow Y = 2$

dire $X = 2$

b) $Y^2 - 4X - 4Y + 16 = 0$

$YY_0 - 2(X + X_0) - 2Y - 2Y_0 + 16 = 0$

$(X_0, Y_0) = (4, 4)$

$4Y - 2X - \cancel{8} - 2Y - \cancel{8} + 16 = 0$

$2Y - 2X + \cancel{16} = 0$

$Y - X + \cancel{8} = 0$

Name *Bob Marks* Box No *156*

C. Given the conic $y^2 - 4x - 4y + 16 = 0$:

a. Sketch the curve, showing both the original and <u>translated</u> axes (properly labeled!), and indicate the important points and/or lines. *??*

(see above)

b. Find the equation of the tangent line at the point (4,4). [You may use calculus methods here if you choose.]

(see above)

c. Find the equation(s) of the tangent line(s) passing through the point (-5,4)

$Y^2 - 4X - 4Y + 16 = 0$

$Y_0^2 - 4X_0 - 4Y_0 + 16 = 0$

$YY_0 - 2X - 2X_0 - 2Y - 2Y_0 + 16 = 0$

$(X, Y) = (-5, 4)$

$Y_0^2 - 4(Y_0 + 9) - 4Y_0 + 16 = 0$

$4Y_0 + 10 - 2X_0 - 8 - 2Y_0 + 16 = 0$

$Y_0^2 - 4Y_0 - 36 - 4Y_0 + 16 = 0$

$2Y_0 - 2X_0 + 18 = 0$

$Y_0 - X_0 + 9 = 0$

$X_0 = Y_0 + 9$ ✓

$Y_0^2 - 8Y_0 - 20 = 0$

$(Y_0 - 10)(Y_0 + 2) = 0$

$Y_0 = (10 \text{ or } -2)$ ✓

$X_0 = (19 \text{ or } 7)$ ✓

t_1 thru (19,10) (-5,4)

$m_1 = \frac{6}{24} = \frac{1}{4}$

$Y - 4 = \frac{1}{4}(X + 5)$

$6Y - 24 = X + 5$

$6Y - X - 29 = 0$

t_2 thru (7,-2) (-5,4)

$m_2 = \frac{6}{-12} = -\frac{1}{2}$

$Y - 4 = -\frac{1}{2}(X + 5)$

$8 - 2Y = X + 5$

$X + 2Y - 3 = 0$ ✓

Name ______________________ Box No ____

D. a. Write the equation of the family of circles which have their centers (h,k) on the curve $y = x^2$ and which are tangent to the x-axis.

b. For what values of h (if any) will these circles intersect the y-axis?

E. Using rotation of axes to reduce $xy = 4$ to the standard form of some conic sect[illegible] determine from the rotated equation the vertices, foci, excluded areas, directrices, asymptotes (if any), and sketch the curve showing both pairs of axes properly labeled.

Name_________________________ Box No_______

F. Given that $A = 2e^x$, $B = 3e^y$, and $A \circ B = \dfrac{6}{e^y}$, describe completely the locus in the xy-plane of y as a function of x. [I.e., determine the function and sketch it.]

G. Given the simultaneous equations

$$\begin{aligned} x - 3y + 2z &= 1 \\ 2x - y + 3z &= 9 \\ x + y + z &= 6 \end{aligned}$$

use Cramer's rule to find the values of x, y, and z, and check your results by substitution.

Calculus

$$x^2 + y^2 = 1 \quad \text{Implicit function}$$

$$\left.\begin{array}{l} y_1 = +\sqrt{1-x^2} \\ y_2 = -\sqrt{1-x^2} \end{array}\right\} \text{Explicit functions}$$

Defn:
1) Average speed = $\dfrac{\text{distance covered}}{\text{time required}}$
2) Inst. speed = limit of average speed as time shrinks to zero

i.e. $V = \lim\limits_{\Delta t \to 0} \dfrac{\Delta s}{\Delta t}$

PROB)

$s = 16t^2$ at $t = 2$ sec.

Let $t = 2 + \Delta t$ be an arbitrary time near $t = 2$ sec

$\Delta s + s = 16(t + \Delta t)^2$

Ave $V = 64 \frac{ft}{sec}$

Defn. of Derivative

$y = f(x)$

$y + \Delta y = f(x + \Delta x)$

$\Delta y = f(x + \Delta x) - f(x)$

$\dfrac{\Delta y}{\Delta x} = \dfrac{f(x + \Delta x) - f(x)}{\Delta x}$

$\lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = \lim \dfrac{f(x + \Delta x) - f(x)}{\Delta x} = F(x)$ = new function

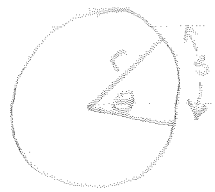
Defn.
1) Heuristic - based on intuitive reasoning
2) Inductive reasoning - Broad to specific
3) Deductive reasoning - specific to broad
4) A Priori - cause & effect - Before-hand knowledge
5) Generic Point - Wide - general aplication

antiderivitive

$Y = kx + C_1$

$Y = k\frac{X^2}{2} + C_1X + C_2$

Radians



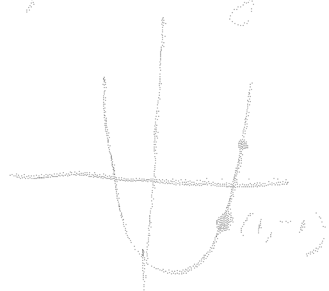
$S = r\theta$

$\frac{dS}{d\theta} = r$

$\frac{\Delta S}{\Delta t} = \frac{r\Delta\theta}{\Delta t} = r\left(\frac{\Delta\theta}{\Delta t}\right)$

$\frac{dS}{dt} = r\frac{d\theta}{dt}$

Ex) find tang. line to $Y = 3X^2 - 4$ at $(1, -1)$



$\frac{dY}{dX} = 6X$

$X = 1$

$\frac{dY}{dX} = 1$

Tang. line

$(Y + 1) = 6(X - 1)$

$Y = 6X - 7$

PROOFS:

1) if $Y = X^n$ then $\frac{dY}{dX} = nX^{n-1}$

n is ≥ 0

$Y = X^n$

$Y + \Delta Y = (X + \Delta X)^n$

$\Delta Y = (X + \Delta X)^n - X^n$

$= [(X + \Delta X) - X]\underbrace{[(X+\Delta X)^{n-1} + (X+\Delta X)^{n-2}X \ldots + (X^{n-1})]}_{n \text{ terms}}$

$\frac{\Delta Y}{\Delta X} = (X+\Delta X)^{n-1} + (X+\Delta X)^{n-2}X \ldots + X^{n-1}$

$dY/dX = nX^{n-1}$

2) $Y = X^{\frac{P}{Q}}$ where $\frac{P}{Q} \neq 0$

$$\Delta Y = (X + \Delta X)^{\frac{P}{Q}} - X^{\frac{P}{Q}}$$

$$\frac{\Delta Y}{\Delta X} = \frac{(X + \Delta X)^{\frac{P}{Q}} - X^{\frac{P}{Q}}}{(X + \Delta X) - X}$$

$$= \frac{(X + \Delta X)^{\frac{P}{Q}} - X^{\frac{P}{Q}}}{(X + \Delta X)^{\frac{Q}{Q}} - X^{\frac{Q}{Q}}}$$

let $S_0 = X^{\frac{1}{Q}}$ $\quad S = (X + \Delta X)^{\frac{1}{Q}}$

$$\frac{\Delta Y}{\Delta X} = \frac{S_0^P - S^P}{S_0^Q - S^Q} = \frac{(S_0 - S)\overbrace{(S_0^{P-1} + S_0^{P-2} S \cdots S^{P-1})}^{P \text{ terms}}}{(S_0 - S)\underbrace{(S_0^{Q-1} + S_0^{Q-2} S \cdots S^{Q-1})}_{Q \text{ terms}}}$$

as $\Delta X \to 0$, $S \to S_0$

$$\frac{dY}{dX} = \frac{P}{Q}\frac{S_0^{P-1}}{S_0^{Q-1}} = \frac{P}{Q}\left(\frac{X^{\frac{P-1}{Q}}}{X^{\frac{Q-1}{Q}}}\right) = \frac{P}{Q} X^{\frac{P}{Q} - 1}$$

3) $Y = \frac{1}{X} = X^{-1}$

$$Y + \Delta Y = \frac{1}{X + \Delta X}$$

$$\Delta Y = \frac{1}{X + \Delta X} - \frac{1}{X} = \frac{X - (X + \Delta X)}{X(X + \Delta X)} = \frac{-\Delta X}{X(X + \Delta X)}$$

$$\Delta Y / \Delta X = \frac{-1}{X(X + \Delta X)}$$

$$\frac{dY}{dX} = \frac{-1}{X^2} = -X^{-2}$$

or

$Y = X^{-m}$ where $m > 0$

$$Y + \Delta Y = \frac{1}{(X + \Delta X)^m}$$

$$\Delta Y = \frac{1}{(X + \Delta X)^m} - \frac{1}{X^m}$$

$$= \frac{X^m - (X + \Delta X)^m}{X^m (X + \Delta X)^m}$$

$$\frac{\Delta Y}{\Delta X} = \frac{-\overbrace{[(X + \Delta X)^{m-1} + (X + \Delta X)^{m-2} X \cdots + X^{m-1}]}^{m \text{ terms}}}{X^m (X + \Delta X)^m}$$

$$\frac{dY}{dX} = \frac{-m(X)^{m-1}}{X^{2m}} = -m X^{-m-1}$$

4) $Y = U(x) + V(X)$

$$Y + \Delta Y = (U + \Delta U) + (V + \Delta V)$$

$$\Delta Y = \Delta U + \Delta V$$

$$\frac{\Delta Y}{\Delta X} = \frac{\Delta U}{\Delta X} + \frac{\Delta V}{\Delta X}$$

$$\frac{dY}{dX} = \frac{dU}{dX} + \frac{dV}{dX}$$

5) $Y = U(X)\ V(X)$

$$Y + \Delta Y = (U + \Delta U)(V + \Delta V) = UV + U\Delta V + V\Delta U + \Delta U \Delta V$$

$$\Delta Y = U\Delta V + V\Delta U + \Delta U \Delta V$$

$$\frac{\Delta Y}{\Delta X} = U\frac{\Delta V}{\Delta X} + V\frac{\Delta U}{\Delta X} + \Delta U \frac{\Delta V}{\Delta X}$$

$$\Downarrow$$

$\Delta U \to 0$ as $\Delta X \to 0$

$$\therefore\ \Delta U \frac{\Delta V}{\Delta X} = 0$$

$$\frac{dY}{dU} = U\frac{dV}{dX} + V\frac{dU}{dX}$$

6) $Y = \dfrac{U(X)}{V(X)}$

$$Y + \Delta Y = \frac{U + \Delta U}{V + \Delta V}$$

$$\Delta Y = \frac{U + \Delta U}{V + \Delta V} - \frac{U}{V} = \frac{V(U + \Delta U) - U(V + \Delta V)}{V(V + \Delta V)}$$

$$= \frac{VU + V\Delta U - UV - U\Delta V}{V(V + \Delta V)}$$

$$\frac{\Delta Y}{\Delta X} = \frac{V\frac{\Delta U}{\Delta X} - U\frac{\Delta V}{\Delta X}}{V(V + \Delta V)}$$

$$\frac{dY}{dX} = \frac{VU' - UV'}{V^2}$$

7) $Y = \sin X$

$$Y + \Delta Y = \sin(X + \Delta X)$$

$$\frac{\Delta Y}{\Delta X} = \frac{\sin(X + \Delta X) - \sin X}{\Delta X}$$

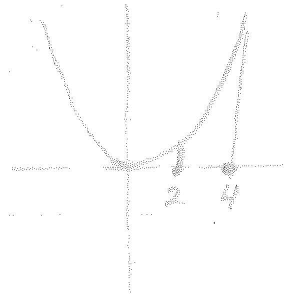
$$\frac{2\cos\left(X + \frac{\Delta X}{2}\right)\sin\frac{\Delta X}{2}}{\Delta X} = \cos\left(X + \frac{\Delta X}{2}\right)\left[\frac{\sin\frac{\Delta X}{2}}{\frac{\Delta X}{2}}\right]$$

$$= \cos X$$

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$-\sec^2 x$
$\cot x$	$-\csc^2 x$
$\sec x$	$\sec x \tan x$
$\csc x$	$-\csc x \cot x$

Pg 141

1)



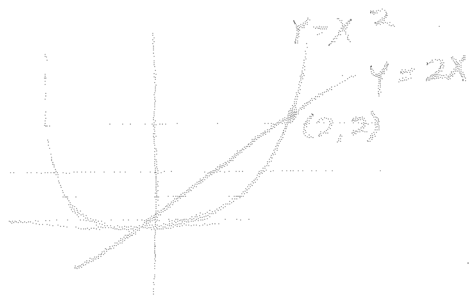
$Y = 3X^2$ $\qquad$ $Y = 5X^2$

$Y = X^3$ $\qquad$ $Y = \frac{5}{3}X^3$

$64 - 8 = 56$ $\qquad$ $\frac{5}{3}(56) = \frac{280}{3} = 93\frac{1}{3}$

$93\frac{1}{3} - 56 = 37\frac{1}{3}$

3)



$Y = X^2$ $\qquad$ $Y = 2X$

$Y = \frac{1}{3}X^3$ $\qquad$ $Y = X^2$

$\frac{8}{3}$ $\qquad$ $4 = \frac{12}{3}$

$\frac{12}{3} - \frac{8}{3} = \frac{4}{3}$

5) $Y^2 = 16X$ $\qquad$ $Y^2 = X^3$

$Y = 4X^{\frac{1}{2}}$ $\qquad$ $Y = X^{\frac{3}{2}}$

$4X^{\frac{1}{2}} = X^{\frac{3}{2}}$

$4 = X$

8 $\qquad$ $16\sqrt{2}$

1) a) $\acute{y} = 4(x^2+1)^3\,2x$

$u^n \frac{du}{dx}$

$u = x^2+1$

$\frac{du}{dx} = 2x$

$\acute{y} = 4\frac{1}{4}(u)^4 \frac{1}{2x}\,2x$

$= (x^2+1)^4 + C$

c) $\acute{y} = (x^2+1)^3 x$

$u = x^2+1$

$\frac{du}{dx} = 2x$

$y = \frac{1}{4}(x^2+1)^4 \frac{1}{2x} x$

$= \frac{1}{8}(x^2+1)^4 + C$

d) $\acute{y} = (x^2+7x)^3(2x+7)$

$u = x^2+7x$

$\frac{du}{dx} = 2x+7$

$y = \frac{1}{4}(x^2+7x)^4 + C$

g) $\acute{y} = 2x(x^2+6)^{-5}$

$u = x^2+6$

$\frac{du}{dx} = 2x$

$y = \frac{1}{2x}(2x)\,\frac{1}{-4}(x^2+6)^{-4} + C$

$= -\frac{1}{4}(x^2+6)^{-4} + C$

i) $\acute{y} = (1+5x)^{\frac{1}{2}}$

$u = 1+5x$

$\frac{du}{dx} = 5$

$y = \frac{1}{5}\,\frac{2}{3}(1+5x)^{\frac{3}{2}}$

$= \frac{2}{15}(1+5x)^{3/2} + C$

7) $y = x(x^2+4)^{-\frac{3}{2}}$

$\frac{du}{dx} = 2x$

$y = \frac{1}{2} \cdot \frac{-1}{2}(x^2+4)^{-\frac{1}{2}}$

$= -\frac{1}{4}(x^2+4)^{-\frac{1}{2}} + C$

8) $y = x(4+5x)^{\frac{1}{2}}$

$u = 4+5x$

$x = \frac{u-4}{5}$

$y = \frac{u-4}{5} u^{\frac{1}{2}}$

$= \frac{u^{3/2} - 4u^{\frac{1}{2}}}{5}$

$= \frac{(4+5x)^{\frac{3}{2}} - 4(4-5x)^{\frac{1}{2}}}{5}$

Differentiate each of the following and simplify the results:

1. $y = (2x-1)^4 \Rightarrow \frac{dy}{dx} = 4(2x-1)^3(2) = 8(2x-1)^3$

2. $y = (1-x^2)^4 \Rightarrow \frac{dy}{dx} = 4(1-x^2)^3(-2x) = -8x(1-x^2)^3 = 8x(x^2-1)^3$

3. $y = \frac{x-1}{\sqrt{x+1}} \Rightarrow \frac{dy}{dx} = \frac{(\sqrt{x+1})(1) - (x-1)\left(\frac{1}{2\sqrt{x+1}}\right)}{(x+1)} = \frac{2(x+1)-(x-1)}{2(x+1)^{3/2}} = \frac{x+3}{2(x+1)^{3/2}}$

4. $y = \frac{(x+1)\sqrt{x-1}}{x^2} \Rightarrow y' = \frac{(x^2)\left[(x+1)\cdot\frac{1}{2\sqrt{x-1}} + \sqrt{x-1}\,(1)\right] - \left[(x+1)\sqrt{x-1}\right][2x]}{x^4}$

$= \left\{x\left[\frac{x+1+2(x-1)}{2\sqrt{x-1}}\right] - (x+1)\sqrt{x-1}\right\} \div \left\{2x^3\sqrt{x-1}\right\}$

$= \left\{2x^2+x-2(x+1)(x-1)\right\} \div \left\{4x^3(x-1)\right\} = \frac{x+2}{2x^3\sqrt{x-1}}$

5. $y = \sqrt{1+\sqrt{x}} = \sqrt{u}$ where $u = 1+\sqrt{x}$

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{2\sqrt{u}}\cdot\frac{1}{2\sqrt{x}} = \frac{1}{4\sqrt{1+\sqrt{x}}\cdot\sqrt{x}}$

Using implicit differentiation, show that

a. if $xy - 2x + 6y - 7 = 0$, then $\frac{dx}{dy} = \frac{1}{dy/dx}$

$\left(x\frac{dy}{dx} + y\cdot 1\right) - 2 + 6\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{2-y}{6+x}$

$\left(x\cdot 1 + y\frac{dx}{dy}\right) - 2\frac{dx}{dy} + 6 = 0 \Rightarrow \frac{dx}{dy} = -\frac{x+6}{y-2} = \frac{6+x}{2-y} = \frac{1}{\left(\frac{dy}{dx}\right)}$

b. if $x^2+y^2 = a^2$, then $\frac{d^2x}{dy^2} \neq \frac{1}{d^2y/dx^2}$

$2x + 2y\frac{dy}{dx} = 0$

$\Downarrow$

$\frac{dy}{dx} = -\frac{x}{y}$

$\Downarrow$

$\frac{d^2y}{dx^2} = -\frac{\left[y\cdot 1 - x\frac{dy}{dx}\right]}{y^2}$

$= -\frac{1}{y^2}\left(y + \frac{x^2}{y}\right) = -\frac{x^2+y^2}{y^3}$

$= -\frac{a^2}{y^3}$

and $2x\frac{dx}{dy} + 2y = 0$

$\Downarrow$

$\frac{dx}{dy} = -\frac{y}{x}$

$\frac{d^2x}{dy^2} = -\frac{\left[x\cdot 1 - y\frac{dx}{dy}\right]}{x^2}$

$= -\frac{1}{x^2}\left[x - y\left(-\frac{y}{x}\right)\right]$

$= -\frac{1}{x^2}\left(\frac{x^2+y^2}{x}\right) = \dots$

$\therefore \frac{d^2x}{dy^2} \neq \frac{1}{\frac{d^2y}{dx^2}}$

[illegible]

Let $A(x)$ = area from $x = -1$ to vertical at x. Then

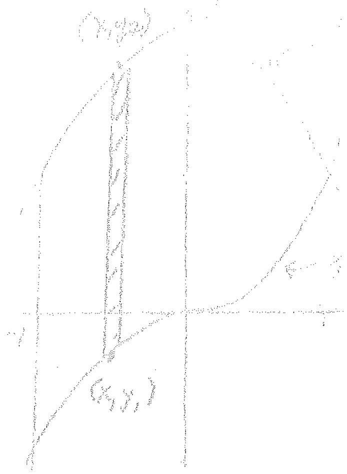
$$\frac{dA}{dx} = [y_2 - y_1] = (2 - x^2) - (x^3)$$

$$A(x) = 2x - \frac{x^3}{3} - \frac{x^4}{4} + C$$

$$0 = A(-1) = -2 + \frac{1}{3} - \frac{1}{4} + C$$

$$A(1) = 2 - \frac{1}{3} - \frac{1}{4} + C$$

$$\Rightarrow A = 4 - \frac{2}{3} + 0 = \frac{10}{3}$$



Integrate each of the following:

a. $\frac{dy}{dx} = (2x+1)^5 = \frac{1}{2}\left[u^5 \frac{du}{dx}\right] \Rightarrow y = \frac{1}{2}\frac{u^6}{6} + C = \underline{\frac{(2x+1)^6}{12} + C}$

$u = 2x+1$

b. $\frac{ds}{dt} = t\sqrt{t^2+1} = \frac{1}{2}\left[u^{1/2}\frac{du}{dt}\right] \Rightarrow s = \frac{1}{2}\frac{u^{3/2}}{3/2} + C = \underline{\frac{1}{3}(t^2+1)^{3/2} + C}$

$u = t^2;\ \frac{du}{dt} = 2t$

c. $\frac{dy}{dz} = z^2(z+1)^2 = z^4 + 2z^3 + z^2 \Rightarrow y = \frac{z^5}{5} + \frac{z^4}{2} + \frac{z^3}{3} + C$

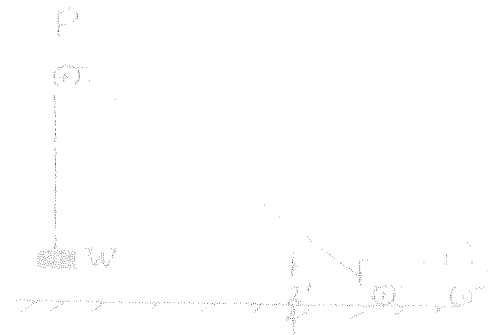
d. $r = \frac{(\theta^{1/3}+3)^3}{\theta^{2/3}} = 3\left[u^3 \frac{du}{d\theta}\right] \Rightarrow r = 3\frac{u^4}{4} + C$

$u = \theta^{1/3} + 3$ $\quad = \underline{\frac{3}{4}(\theta^{1/3}+3)^4 + C}$

$\frac{du}{d\theta} = \frac{1}{3\theta^{2/3}}$

[illegible]

[illegible]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[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

Extra Problems on Related Rates

1) $\frac{dV}{dt} = 2$ $\qquad\qquad$ $\frac{dA}{dt}$

$V = \frac{4}{3}\pi r^3$ $\qquad\qquad$ $A = 4\pi r^2$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$ $\qquad\qquad$ $\frac{dA}{dr} = 8\pi r$

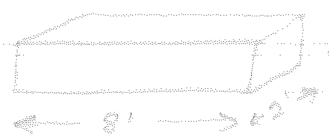
$2 = 4\pi r^2 \frac{dr}{dt}$ $\qquad\qquad$ $\frac{dA}{dt} = 8\pi r \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{1}{2\pi r^2}$ $\qquad\qquad$ $= \frac{4}{r}$

$r = 12$

$\frac{dA}{dt} = \frac{1}{3}$

4)



$\frac{dV}{dt} = 2$ $\qquad\qquad$ $\frac{dh}{dt}$

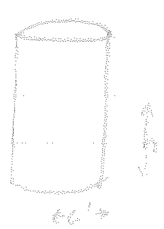
$V = 16h$

$h = \frac{V}{16}$

$\frac{dh}{dV} = \frac{1}{16}$

$\frac{dh}{dt} = \frac{1}{16}\frac{dV}{dt} = \frac{1}{8}$ ft/min

5)



$\frac{dV}{dt} = 8$ $\qquad\qquad$ $\frac{dh}{dt}$

$V = 36\pi h$

$h = \frac{V}{36\pi}$

$\frac{dh}{dV} = \frac{1}{36\pi}$

$\frac{dh}{dt} = \frac{1}{36\pi}\frac{dV}{dt} = \frac{2}{9\pi}$

6)

$\frac{ds}{dt} = 2$ $\qquad\qquad$ $\frac{dh}{dt}$

$s^2 + h^2 = 400$

$h = (400 - s^2)^{\frac{1}{2}}$

$\frac{dh}{ds} = \frac{1}{2}(400 - s^2)^{-\frac{1}{2}} \cdot -2s$

$\frac{dh}{dt} = -2s(400 - s^2)^{-\frac{1}{2}}$

$s = 12$

$\frac{dx}{dt} = \frac{-24}{16} = -\frac{3}{2}$

MATH [illegible]

[illegible] Quiz II 11 Oct 60
(50 pts)

1. Define the following:

a. Tangent line to the curve $y = f(x)$ at a point (x_0, y_0) on the curve. is that line (if any) thru (x_0, y_0) having as its slope the limit of the slopes of secant lines thru (x_0, y_0) and $(x_0+\Delta x, y_0+\Delta y)$ as $\Delta x \to 0$.

b. Slope of the curve $y = f(x)$ at (x_0, y_0) on the curve. (NOTE: $\left.\frac{dy}{dx}\right|_{(x_0, y_0)}$ is a formula for, but not the definition of the slope of the curve at (x_0, y_0).)

The slope of the curve at (x_0, y_0) is defined as the slope of its tangent line at that point.

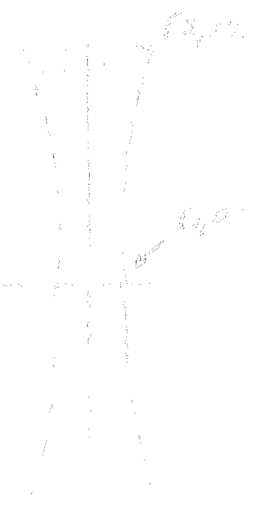
2. a. Write an equation of the tangent line to the curve $y = x^2$ at the point (x_1, y_1) on the curve.

$\frac{dy}{dx} = 2x \Rightarrow m = 2x_1$, so $(y - y_1) = (2x_1)(x - x_1)$ or $y = y_1 + 2x_1 x - 2x_1^2$. But (x_1, y_1) is on the curve, so $y_1 = x_1^2$ and the tangent line is
$y = 2x_1 x - x_1^2$

b. Let (x_1, y_1) and (x_2, y_2) be different points on $y = x^2$. Find x_1 in terms of x_2 so that the tangent lines at these two points will be perpendicular. Are there any restrictions on the points?

slope at $(x_1, y_1) = 2x_1$ & slope at (x_2, y_2) is $2x_2$, so tan. lines will be $\perp$ if $2x_1 = -\frac{1}{2x_2}$ or $x_1 = -\frac{1}{4x_2}$.
This is valid for $x_1 \neq 0$ and $x_2 \neq 0$.

3. A round column 20 ft. long is to be made with a 2 ft. radius at each end, a 1 ft. radius at the middle, and having as its cross-section parallel to the axis parabolic curves (see sketch). Write an equation for this parabola.



$y^2 = \cancel{\quad} = k_1 x + k_2$ (by symmetry about x-axis).

$(1,0) \Rightarrow k_1 + k_2 = 0$
$(2,10) \Rightarrow 2k_1 + k_2 = 100$ $\Big\} \Rightarrow k_2 = -k_1$, $k_1 = 100$

$\therefore y^2 = 100(x-1) \Rightarrow Y^2 = 100X - 1$

[illegible]

[illegible] $(2b-4)+2 = 2b-2$

[illegible] $2(a+2)-4 = 2a$

[illegible] $(\sqrt{2}+2)(2\sqrt{2}-4) = 4-8 = -4$

[illegible] x^2 [illegible] (If $x+1=y$, $x=y-1$, so

[illegible] $(y-1)^2+3(y-1)+7 = y^2+y+5$

4. [illegible] constant, determine [illegible] values of [illegible] where a [illegible] non [illegible] constant.

$(x+a)^2+(x+a)+C = x^2+x+C$

$x^2+(2a+1)x+(a^2+a+C) = x^2+x+C$

$\therefore\ 2ax+a^2+a=0$, and since $a \neq 0$ (), we then must have

$2x+(a+1)=0$

or simply

$\underline{x = -\dfrac{a+1}{2}}.$

$$\Rightarrow \frac{x}{x+1} = \pm 2 .$$

so either $\frac{x}{x+1} = 2 \Rightarrow x = 2(x+1) \Rightarrow x+2=0 \Rightarrow \underline{x=-2}$

or $\frac{x}{x+1} = -2 \Rightarrow x = -2(x+1) \Rightarrow 3x = -2 \Rightarrow \underline{x = -\frac{2}{3}}$

Either

$$\frac{x+1}{x-1} > 3 \Rightarrow \left\{\begin{array}{l} x+1 > 3(x-1) \text{ for } (x-1)>0 \\ \underline{\text{or}} \\ x+1 < 3(x-1) \text{ for } (x-1)<0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} 2x < 4 \text{ and } x>1 \Rightarrow \underline{1<x<2} \\ \underline{\text{or}} \\ 2x > 4 \text{ and } x<1 \text{ (impossible)} \end{array}\right.$$

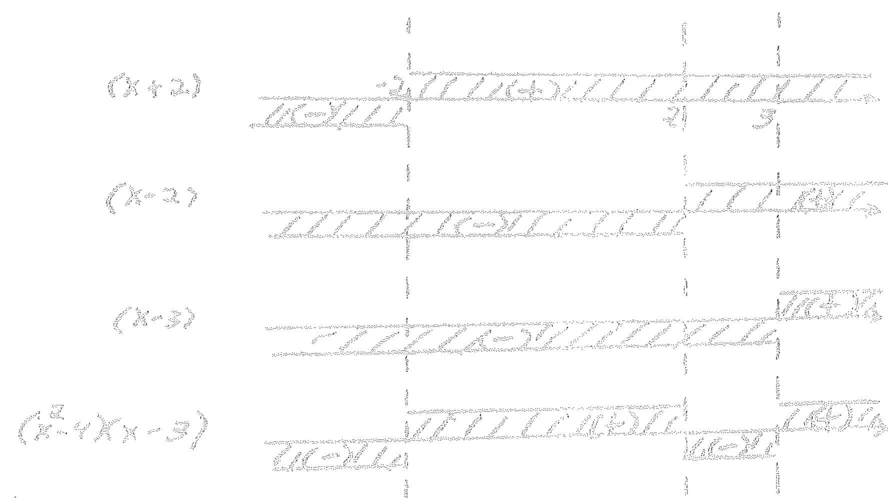
or

$$\frac{x+1}{x-1} < -3 \Rightarrow \left\{\begin{array}{l} x+1 < -3(x-1) \text{ for } (x-1)>0 \\ \underline{\text{or}} \\ x+1 > -3(x-1) \text{ for } (x-1)<0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} 4x < 2 \text{ and } x>1 \text{ (impossible)} \\ \underline{\text{or}} \\ 4x > 2 \text{ and } x<1 \Rightarrow \underline{\tfrac{1}{2}<x<1} \end{array}\right.$$

So the solution set is $x \in (\frac{1}{2}, 1) \cup (1, 2)$.

[In other notation, either $\frac{1}{2} < x < 1$ or $1 < x < 2$.]

Since $x^2+1>0$ for all real values of x, then we must have $(x^2-4)(x-3)>0$ or, on factoring, $(x+2)(x-2)(x-3)>0$. Then the sign diagrams are:

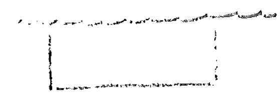


So the solution set is $x \in (-2, 2) \cup (3, +\infty)$

[or: $-2 < x < 2$ or $3 < x < +\infty$].

ASSORTED PROBLEMS FOR YOUR CONSIDERATION

1. A man with 300 yards of fencing wishes to enclose a rectangular area as large as possible along the bank of a straight river. What dimensions should he use?

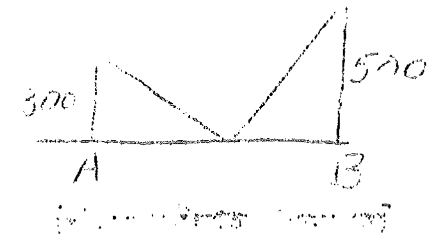


[75 x 150]

2. What positive number plus its reciprocal gives the least sum?

[1]

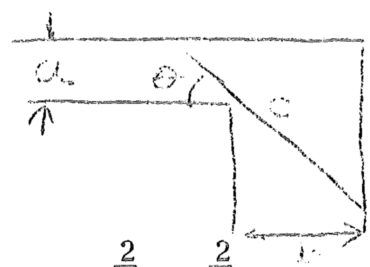
3. Two houses are 300 and 500 yards from a straight power line. Where should they attach to the power line to make the total length of cable a minimum?



4. A cylindrical boiler is to contain 1000 cu. ft. What are the most economical dimensions?

[radius = $10/\sqrt[3]{2\pi}$]

5. Find the narrowest width for b in order that the beam of length c can be gotten round the corner. Neglect the thickness of the beam.



[$b = (c^{\frac{2}{3}} - a^{\frac{2}{3}})^{3/2}$]

6. Radiant heat from a point source varies inversely as the square of the distance and directly as the intensity of the source. If two sources at O_1 and O_2 a distance a apart, have intensities c_1 and c_2, what point between them is coolest?

[at a distance from $O_1 = \dfrac{a c_1^{\frac{1}{3}}}{c_1^{1/3} + c_2^{1/3}}$]

7. Find the cylindrical can with open top that has least total survace for a given volume.

[radius = height]

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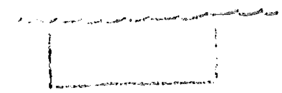

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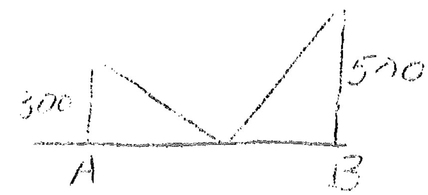

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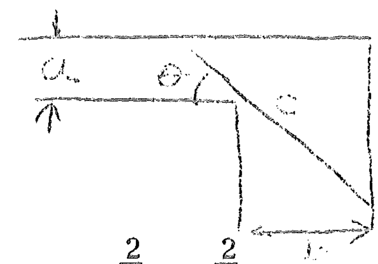

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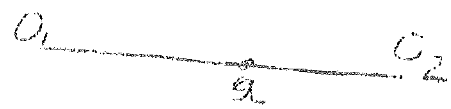
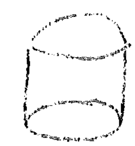


[at a distance from $O_1 = \dfrac{ac_1^{\frac{1}{3}}}{c_1^{1/3} + c_2^{1/3}}$]

7. Find the cylindrical can with open top that has least total survace for a given volume.

[radius = height]

8. A gas tank of volume V is to be made in the shape of a cylinder surmounted by a hemisphere. What should be its proportions for minimum material?



[radius = height]

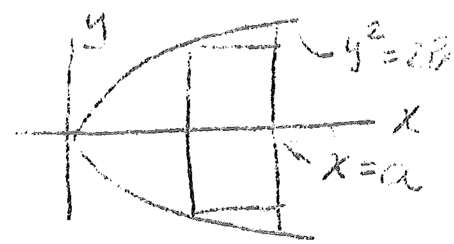
9. A wire of length c is cut in two. One piece is bent to form a square and the other piece to form a circle. (a) How should it be cut to enclose the minimum area? (b) Maximum?

[(a) perimeter of square = $\frac{4}{5}c$
(b) only the circle. $4+\pi$)]

10. A water tank is to have a square base and open top and contain 1000 gallons. If the base is twice as costly as the sides what proportions give minimum material cost? [depth = side of base]

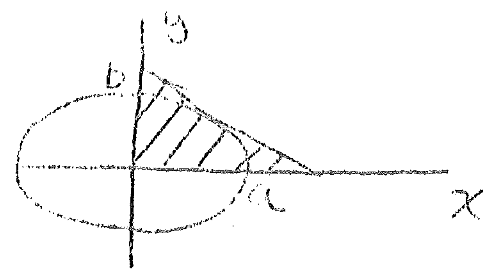
11. At what point P does the rectangle with a vertex at P have maximum area?

[$(\frac{a}{3}, \sqrt{\frac{2aP}{3}})$]



12. A Norman window has the shape of a rectangle surmounted by a semicircle. For a given perimeter what proportions give greatest area? [radius = height rectangle]

13. What is the minimum area of the triangle formed by the axes and the tangent line to the ellipse with semi-axes a and b.
[area = 2ab]



15. A ball is thrown vertically upward and reaches a height of 100 ft. If the angle of elevation of the sun is 60^0 how fast is the shadow of the ball moving 2 seconds after it begins to fall?

[$\frac{64}{\sqrt{3}}$ ft/sec]

16. Find the dimensions of the rectangle of maximum area than can be inscribed in a semi-circle of radius r.

17. A 24 foot ladder leans against a high wall. If the foot of the ladder is pulled away from the base od the wall at the rate of 6 ft/min., how fast is the top moving when the foot is 8 feet from the base of the wall? [descending $\frac{3}{\sqrt{2}}$ ft/min]

18. A man on a pier pulls in a rope attached to a small boat at the rate of 1 foot per second. If his hands are 10 feet above the place where the rope is attached, how fast is the boat approaching the pier when there is 20 feet of rope out?

[1.2 ft/sec]

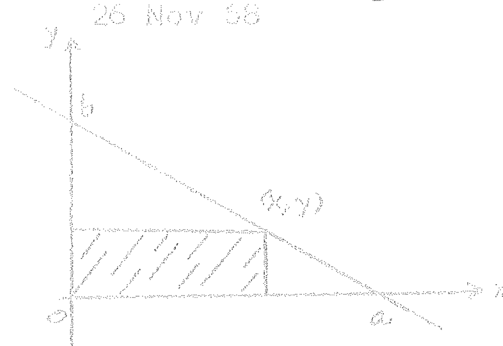
19. A cylindrical tank with axis horizontal has a diameter of 6 feet and a length of 15 feet. It is partly full of oil to a depth of 4 feet when a leak starts to drain off the oil at the rate of 10 cu ft/min. How fast is the level falling?

[1.4 in/min]

20. A ball of radius a rests in a hemispherical bowl of radius 2a containing water. Show that $\dot{V} = 4\pi\, a\, h\, \dot{h}$

(AR1) Test ___ (63)

Name Bob Marks
26 Nov 58



1. Determine the maximum area possible for a rectangle having one corner at (0,0) and the opposite corner on the line (x/a)+(y/b)=1 and sides parallel to the coordinates axes, as shown in the sketch to the right:

a. By using the "explicit" method

$\cancel{A = XY \quad \frac{X}{A}+\frac{Y}{B}=1}$

$\cancel{\frac{Y}{B} = 1-\frac{X}{A}}$

$\cancel{Y = B - \frac{BX}{A}}$

$\cancel{A = X\left(B-\frac{BX}{A}\right) = X\left(\frac{AB-BX}{A}\right)}$

$\cancel{\frac{dA}{dX} = B - \frac{2B}{A}X = 0}$

$\cancel{X = \frac{A}{2}}$

-20

b. By using the "~~implicit~~ explicit" method

$A = XY \qquad \frac{X}{A}+\frac{Y}{B}=1$

$\frac{X}{A} = 1-\frac{Y}{B} = \frac{B-Y}{B}$

$X = \frac{AB-AY}{B}$

$A = \frac{AB-AY}{B}Y$

$= \frac{ABY-AY^2}{B} = AY - \frac{A}{B}Y^2$

$\frac{dA}{dY} = A - \frac{2AY}{B} = 0$

$A = \frac{2AY}{B}$

$B = 2Y$

$Y = \frac{B}{2}$

$\frac{Y}{B} = 1-\frac{X}{A} = \frac{A-X}{A}$

$Y = \frac{BA-BX}{A}$

$A = \frac{BA-BX}{A}X = BX - \frac{B}{A}X^2$

$\frac{dA}{dX} = B - \frac{2B}{A}X = 0$

$X = \frac{a}{2}$

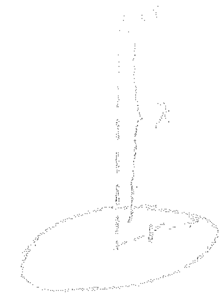
$\cancel{A = \frac{\frac{a}{2}}{a}} + \frac{\frac{b}{2}}{b} = 1$

$\frac{1}{2}+\frac{1}{2}=1$

$A = XY = \frac{ab}{4}$

-7

... coil of radius r, ... force $F = \dfrac{kx}{(x^2+r^2)^{5/2}}$ (k=constant)
on a small magnet located at a distance x above the center of the coil. Determine the maximum value for F (if any).



$$F = \frac{kx}{(x^2+r^2)^{\frac{5}{2}}}$$

$$\frac{d}{dx}(x^2+r^2)^{\frac{5}{2}}$$

$$U = x^2+r^2 \qquad F = U^{\frac{5}{2}}$$

$$\frac{d(x^2+r^2)^{\frac{5}{2}}}{dx} = \frac{dF}{dU}\frac{dU}{dx}$$

$$= \frac{5}{2}(x^2+r^2)^{\frac{3}{2}}2x$$

$$= 5x(x^2+r^2)^{\frac{3}{2}}$$

$$\frac{dF}{dx} = \frac{k(x^2+r^2)^{\frac{5}{2}} - 5x(x^2+r^2)^{\frac{3}{2}}kx}{(x^2+r^2)^5} = 0$$

$$= k(x^2+r^2)^{\frac{5}{2}} - 5kx^2(x^2+r^2)^{\frac{3}{2}}$$

$$= k(x^2+r^2)^{\frac{3}{2}}\left[(x^2+r^2) - 5x^2\right]$$

$$x^2+r^2 > 0 \Rightarrow (x^2+r^2) - 5x^2 = 0$$

$$x^2+r^2-5x^2 = 0$$

$$r^2 = 4x^2$$

$$r = 2x$$

$$F = \frac{kx}{(x^2+4x^2)^{\frac{5}{2}}}$$

$$= \frac{kx}{(5x^2)^{\frac{5}{2}}}$$

$$= \frac{kx}{(5)^{\frac{5}{2}}x^5}$$

$$F_{max} = \frac{k}{25\sqrt{5}x^4}$$

3. Sand falling from a chute forms a conical pile whose altitude is always equal to 2/3 of the diameter of the base. If the height is observed to be increasing at the rate of 1 ft/min when the height is 4 ft, how fast (in ft^3/min) is sand flowing from the chute?

$$h = \frac{2}{3}(2r) \qquad \frac{dh}{dt} = 1$$

$$h = \frac{4}{3}r$$

$$\frac{dv}{dt} = ?$$

$$\frac{dh}{dr} =$$

$$r = \frac{3}{4}h$$

$$\frac{dr}{dh} = \frac{3}{4}$$

$$\frac{dr}{dt} = \frac{3}{4}\frac{dh}{dt} \qquad \frac{dh}{dt} = 1$$

$$\frac{dr}{dt} = \frac{3}{4}$$

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi r^2\left(\frac{4}{3}r\right)$$

$$V = \frac{4}{9}\pi r^3$$

$$\frac{dV}{dt} = \frac{4}{3}\pi r^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{3}{4}$$

$$\frac{dv}{dt} = \pi\left(\frac{3}{4}h\right)^2$$

$$= \pi\left(\frac{3}{4}4\right)^2$$

$$\frac{dv}{dt} = 9\pi \frac{FT^3}{MIN}$$

$3/2 \quad \frac{1}{2} \quad \frac{3}{2}$

$(x-1)(x-2)$.

determine the relative max. & min. pt. ... the flex pt. ..., and sketch the curve showing its principal features.

-1

$Y' = 0 \Rightarrow X = \frac{1}{2}$ or $X = 2$

$X = 2 \Rightarrow Y = 0$ - flex pt at $(2,0)$ also at $(1,-1)$ for $X = 2 \Rightarrow Y'' = 0$

$X = \frac{1}{2} \Rightarrow Y = \frac{-27}{16}$ $\quad X = \frac{1}{2} \Rightarrow \ddot{Y} = 9 > 0 \therefore (\frac{1}{2}, \frac{-27}{16})$ is a min

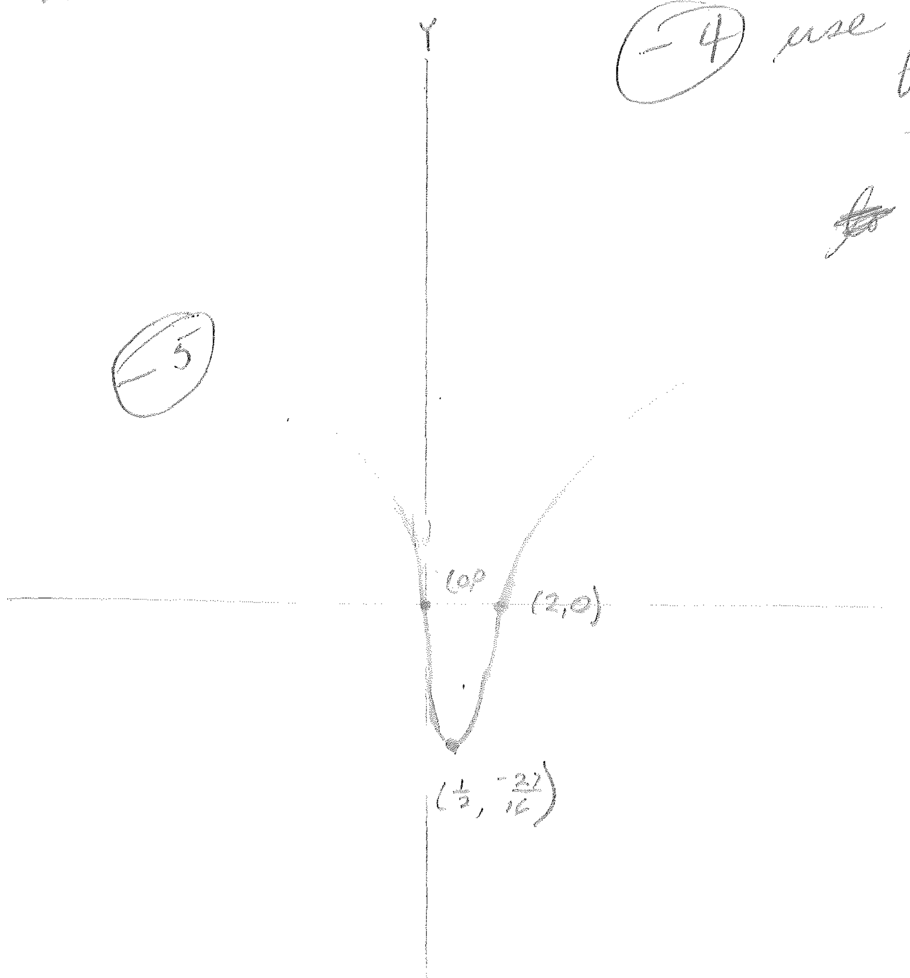
$X = 1 \Rightarrow \dot{Y} = 2 \Rightarrow \ddot{Y} = 0 \quad X = 7 \Rightarrow \dot{Y} < 0$

$X = 0 \Rightarrow Y = 0$
$X = 2 \Rightarrow Y = 0$

-4 use excl. regions to help sketch

~~to~~ find symm., asym., exc. regions

-5



C A L C U L U S   II

FINAL EXAMINATION

March 19, 1969

NAME Bob Marks   BOX 156

INSTRUCTOR Prof. Godert   SECTION C

INSTRUCTIONS: WORK ALL PROBLEMS

(PLEASE DO NOT WRITE BELOW THIS LINE)

1. 8 ~~9~~
2. 9
3. 8
4. 12 +2
5. 2
6. 12 +2
7. 5
8. 3
9. 6

TOTAL 65 + 4 = 69

1. Find $\frac{dy}{dx}$ for each of the following:

(a) $y = 2\sin^{-1}(ax)$

$Y = 2\sin^{-1} U \qquad U = ax$

-2 $\quad \frac{dY}{dX} = \frac{2a}{\sqrt{1-a^2X^2}}$

(b) $y = \log(x^5)$

$\frac{dY}{dX} = \frac{5\cancel{X^4}}{X^5} = \frac{5}{X}$

(c) $y = e^{2x}\cos(3x)$

$\frac{dY}{dX} = -3e^{2X}\sin 3X + 2e^{2X}\cos 3X$

(d) $y = (x)^{\log(x)}$

$\ln Y = \ln X \ln X$

$\ln Y = (\ln X)^2$

-2 $\quad \frac{y'}{y} = \frac{2}{X}\ln X$

$\frac{dy}{dx} = \frac{2y\log x}{X} = \frac{2\ln x}{X}x^{\ln x}$

$\ln Y = (\ln X)^2$

$U = \ln X \qquad \frac{dU}{dX} = \frac{1}{X}$

$\ln Y = U^2 \quad \frac{dY}{dU} = 2U^3$

$\frac{dY}{dX} = \frac{2(\ln X)^3}{X}$

2. Integrate each of the following:

(a) $y = \int \tan^2 x \sec^2 x \, dx$

$dV = \sec^2 x \quad U = \tan^2 x$
$V = \tan x \quad dU = \tan x \sec^2 x dx$

$Y = \tan^3 x - \int \tan^2 x \sec^2 x dx$

$2Y = \tan^3 x$

$Y = \frac{\tan^3 x}{2} + C$

(b) $y = \int x^2 e^{ax^3} dx$

$\cancel{e^{ax^3}}$

$Y = \frac{e^{ax^3}}{3a} + C$

-1

(c) $\frac{dy}{dx} = \cos^2 x = 1 - \sin^2 x$

$U = \sin x \quad dV = -\sin x$
$dU = \cos x dx \quad V = \cos x$

$Y = X + \sin x \cos x - \int \cos^2 x$

$2Y = X + \sin x \cos x$

$Y = \frac{x}{2} + \frac{\sin x \cos x}{2} + C$

(d) $\frac{dy}{dx} = y + 1$

$\frac{dx}{dY} = \frac{1}{Y+1}$

$x = \ln(Y+1) + C$

-2

$e^x = Y + 1 \qquad e^{x-c} = y+1 = e^{-c} \cdot e^x = ae^x$ where $a = e^{-c}$

$-Y = 1 - e^x \qquad y = ae^x - 1$

It is essential to include a constant of integration
absolutely necessary here
Better get into the habit of putting it in everywhere

3. Integrate each of the following:

a) $y = \int \sqrt{4 - x^2}\, dx$

$x = 2\sin\theta \qquad dx = 2\cos\theta\, d\theta$

$Y = \int 2\cos\theta\, dx$

$= 4\int \cos^2\theta\, d\theta$

$4\left(\frac{\theta}{2} + \frac{\cancel{\sin\theta\cos\theta}}{4}\right)$

$2\sin^{-1}\frac{x}{2} + \frac{1}{4}\sin 2\left(\sin^{-1}\frac{x}{2}\right)$ not an acceptable ans. $= 2\sin^{-1}\frac{x}{2} + \frac{x\sqrt{4-x^2}}{2} + C$

$\theta = \sin^{-1}\frac{x}{2}$

$\sin\theta = \frac{x}{2}$

$\cos\theta = \sqrt{4-x^2}$

ALWAYS draw triangle

-1

b) $\frac{dy}{dx} = \frac{1}{\sqrt{1 - x}}$

$Y = -2\sqrt{1-x} + C$

c) $y = \int \frac{x^2}{x^2 + 16}\, dx \quad = 4\int \tan^2\theta\, d\theta = 4\tan\theta - 4\theta + C$

$= x - 4\tan^{-1}\frac{x}{4} + C$

so let $x = 4\tan\theta$

-3

$dv = \frac{x}{x^2+16} \qquad u = x$

$v = \frac{1}{2}\ln(x^2+16) \quad du = dx$

$Y = \frac{1}{2}x\ln(x^2+16) - \frac{1}{2}\int \ln(x^2+16)\,dx$

which can't be done!

d) $\frac{dy}{dx} = \frac{x}{\sqrt{1 + x}}$

$1 + x = u$

$x = u - 1$

$\int \frac{u-1}{u^{\frac{1}{2}}} \quad du = dx$

$u = u - 1$

$du = dv$

$v = 2u^{\frac{1}{2}}$

$Y = 2u^{\frac{1}{2}}(u-1) - 2\int u^{\frac{1}{2}}\,du$

$= 2u^{\frac{1}{2}}(u-1) - \frac{4}{3}u^{\frac{3}{2}}$

$= 2x\sqrt{1+x} - \frac{4}{3}(1+x)^{\frac{3}{2}}$

$Y = 2x\sqrt{1+x} - \frac{4}{3}(1+x)^{\frac{3}{2}} + C$

$= \frac{2}{3}(x-2)\sqrt{1+x} + C$

others may be more insistent about adding the necessary constants!

4. (a) Integrate the following

(i) $y = \int e^{x} \sin 3x \, dx$

$U = \sin 3X \quad dV = e^{X}$

$dU = 3\cos 3X\,dX \quad V = e^{X}$

$Y = e^{X}\sin 3X - \int 3\cos 3X e^{X}$

$U = 3\cos 3X \quad dV = e^{X}$

$dU = -9\sin 3X\,dX \quad V = e^{X}$

$Y = e^{X}\sin 3X - 3e^{X}\cos 3X + 9\int e^{X}\sin 3X\,dX$

$$Y = \frac{e^{X}\sin 3X - 3e^{X}\cos 3X}{10} + C$$

(ii) $y = \int \dfrac{x}{(1+x)(1+x^{2})} \, dx$

$$\frac{X}{(1+X)(1+X^{2})} = \frac{A}{1+X} + \frac{BX+C}{1+X^{2}}$$

$X = A(1+X^{2}) + (1+X)(BX+C)$

$X = i \Rightarrow i = (1+i)(Bi+C) = Bi + C - B + Ci = i$

$X = -1 \Rightarrow A = -\frac{1}{2} \qquad Bi - B = i - Ci - C$

$B = C = \frac{1}{2} \qquad Bi = (1-C)i$

$$Y = -\frac{1}{2}\ln(1+X) + \frac{1}{2}\int\frac{X}{1+X^{2}} + \frac{1}{2}\int\frac{1}{1+X^{2}} = \frac{1}{4}\ln(1+X^{2}) - \frac{1}{2}\ln(1+X) + \frac{1}{2}\tan^{-1}X + C$$

(b) A particle moves along a straight line in accordance with the equation $s = \frac{1}{3} t^{3} - 2t + 3$ where t is measured in seconds and s in feet. Using differentials, find approximately the distance covered in the interval from $t = 2$ to $t = 2.1$ seconds.

$s = \frac{1}{3}t^{3} - 2t + 3$

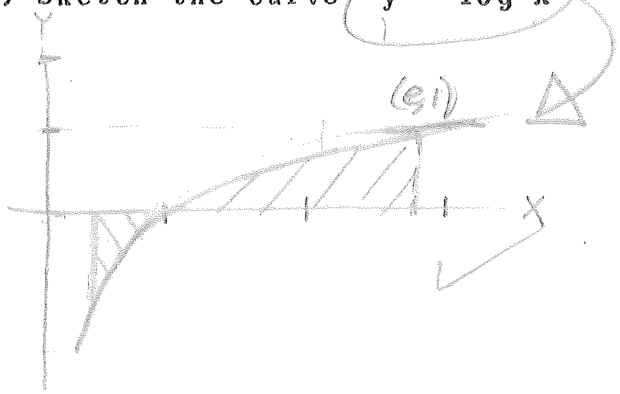
$ds = (t^{2} - 2)\,dt$

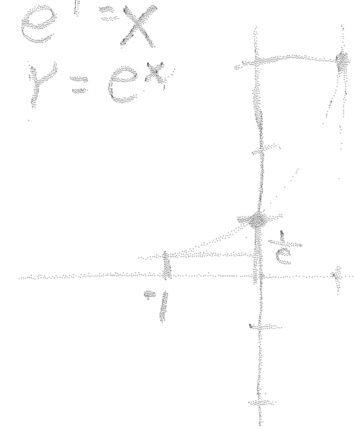
$= (4-2)(.1) = .2\ ft$

5. (a) (i) Sketch the curve $y = \log x$

$e^Y = X$

$Y = e^x$





(ii) Find the total area bounded by the curve $y = \log x$ and the x-axis between $x = \frac{1}{e}$ and $x = e$.

$Y = e^x$

$\int_0^e e^x = (e^e - 1)$

$\int_{\frac{1}{e}}^{-1} = e^{-1} - e^{\frac{1}{e}}$

$2 - \frac{2}{e}$

$e - e^e + 1$

-4

$\cancel{e - e^e} + 1 + \cancel{\frac{1}{e}} - e^{\frac{1}{e}} - \cancel{\frac{1}{e}}$

$A = e - e^e - e^{\frac{1}{e}} + 1$

(b) Use the Mean Value Theorem to prove that $e^x > 1 + x$ for all $x > 0$.

$Y = e^X$

$dY = e^x dX$

$Y = 1 + X$

$d\cancel{Y} = d\cancel{X}$

-4

$f(x) = f(0) + x f'(\bar{x})$

$f'(\bar{x}) = e^{\bar{x}} > 1$

$0 < \bar{x} < x$

$e^x > 1 + x$

C

6. Find the limits of each of the following:

(a) $\lim_{x \to 1} \dfrac{\log(x-1)}{\cot(x-1)} \to \dfrac{\infty}{\infty}$

$\to \dfrac{\frac{1}{x-1}}{-\csc^2(x-1)} \to \dfrac{\infty}{\infty} = \dfrac{-\sin^2(x-1)}{x-1} \Rightarrow \dfrac{0}{0}$

$\dfrac{-\sin 2(x-1)}{1} \Rightarrow 0$ ✓

(b) $\lim_{x \to 0} e^{-1/x^2} \cos x$

$\dfrac{\cos x}{e^{\frac{1}{x^2}}} \to \dfrac{1}{\infty} \longrightarrow 0$ ✓

(c) $\lim_{x \to +\infty} x(e^{1/x} - 1)$

$\dfrac{e^{\frac{1}{x}} - 1}{\frac{1}{x}} \to \dfrac{0}{0}$

$\dfrac{\cancel{-\frac{1}{x^2}} e^{\frac{1}{x}}}{\cancel{-\frac{1}{x^2}}} = 1$ ✓

(d) $\lim_{x \to 0} (e^x + 3x)^{1/x}$

$y = (e^x + 3x)^{\frac{1}{x}}$

$\ln y = \dfrac{\ln(e^x + 3x)}{x} \to \dfrac{0}{0}$

$\dfrac{\frac{e^x + 3}{e^x + 3x}}{1} \to 4$

e^4 ✓

7. (a) A 1/4 pound mass attached to the free end of a spring is in equilibrium position when the spring is stretched 4 inches. Determine the spring constant, k.

$kd = 32m$

$K\left(\frac{1}{3}\right) = 8$

$K = 24$ ✓

(b) The mass of part (a) is raised 2 inches above its equilibrium position and then released. Find the displacement of the mass t seconds after the start of the motion.

$Y = C \sin\left(\sqrt{k/m}\, t + \phi\right)$

$Y = 2 \sin\left(\sqrt{96}\, t + \phi\right)$

−2

$\dot{y} = 2\sqrt{96} \cos\left(\sqrt{96}\, t + \varphi\right)$

$t = 0,\ \dot{y} = 0,\quad \cos\varphi = 0,\ \varphi = \frac{\pi}{2}$

$\therefore\ y = 2 \cos \sqrt{96}\, t$

(c) What is the period of the displacement of part (b)?

$\frac{2\pi}{\sqrt{96}}$

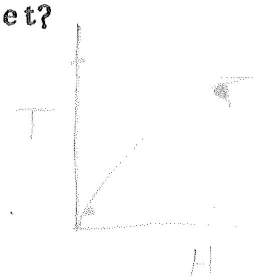
−3

$\frac{dY}{dt} = \sqrt{96} \cos\left(\sqrt{96}\, t + \phi\right)$

8. A heated ball is dropped into a tank of water 200 feet deep. While the ball is descending, the temperature of the ball decreases at a rate (<u>degrees per second</u>) <u>inversely</u> proportional to its depth. The ball drops at a constant vertical velocity of 10 feet per second. If the temperature at a depth of 1 foot is $100^{\circ}C$ and the temperature at a depth of 10 feet is $80^{\circ}C$, what is the temperature at a depth of 100 feet?

(Hint: $\frac{d(temp)}{dt} = \frac{d(temp)}{dh} \cdot \frac{dh}{dt}$

TEMP	h	TIME
80°	10 ft	1 sec
100°	1 ft	$\frac{1}{10}$ sec



$h = 10t$

$\frac{dh}{dt} = 10 \frac{ft}{sec}$

$T = \frac{K}{h}$

$\frac{\Delta T}{\Delta h} \approx \frac{dT}{dh} = \frac{-20}{9}$

$\frac{dT}{dt} = 10 \frac{dT}{dh} = -\frac{K}{h}$

$T = \frac{20}{\log 10} \log h + 100$

$h = 100, T = 60^{\circ}$

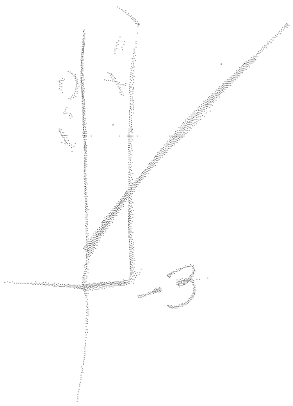
$d = \frac{20}{9} dh$

$dT = 180^{\circ}$

$T = 80^{\circ}C$

9. Consider the region bounded by the curve $y = 1 + x$, the lines $x = 0$ and $x = 1$, and the x-axis.

(a) Divide the interval from 0 to 1 into n equal parts and approximate the area by summing rectangular areas. $\left(1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2}\right)$.



$$S_n = \Delta X(1+\Delta X) + 2\Delta X(1+\Delta X) \ldots n\Delta X(1+\Delta$$

$$(1+2\Delta X)\Delta X + \ldots + (1+n\Delta X)\Delta X$$

$$= (\Delta X + \Delta X^2)(1+2+3+ \ldots K)$$

$$S_n = \frac{3}{2} + \frac{1}{2n}$$

$$n\Delta X = 1$$

$$\Delta X = \frac{1}{n}$$

$$S_n = \left(\frac{1}{K} + \frac{1}{K^2}\right)\left(\frac{K(K+1)}{2}\right) \quad X$$

(b) Take the limit of the expression in (a) as the number of intervals becomes infinite (and each subinterval goes to zero in length) to obtain the exact area.

$$\frac{K^2+2K+1}{2K} \to \frac{2K+2}{2} \to \infty \neq 1.5 \quad \text{or} \quad \frac{K}{2}+1+\frac{1}{2K} \to \infty+1+0 \to \infty$$

see? So your result in (a) must be wrong.

$$\lim S_n \left(\frac{1}{K} + \frac{1}{K^2}\right)\left(\frac{K(K+1)}{2}\right) = \frac{(K+1)(K+1)}{2K} \Rightarrow \lim_{K\to\infty} S_n = 1.5$$

(c) Use the definite integral (i.e., the Fundamental Theorem of Calculus) to obtain the same area.

$$\int_{x_0}^{x_1}(1+X)\,dx = X + \frac{X^2}{2}\Big|_0^1 = 1 + \frac{1}{2} = \frac{3}{2}$$

$$\left(X + \frac{1}{2}X^2\right)\Big|_1^0 = 1\frac{1}{2}$$

Your notation is not in accordance with usual practice

CALCULUS I

FINAL EXAMINATION

Name Bob Marker Box No. 156

Instructor Schmit dt, if you please! Section C

Instructions: Work All 10 Problems.

1.	11
2.	$11\frac{1}{2}$
3.	$9\frac{1}{2}$
4.	10
5.	1
6.	3
7.	4
8.	4
9.	5
10.	4
TOTAL	63/95 = 66%

1. Differentiate each of the following expressions.

(a) $y = \frac{x}{a - x}$

3

$\dot{y} = \frac{(a-x) - x(-1)}{(a-x)^2}$

$\dot{y} = \frac{a - x + x}{(a-x)^2} = \frac{a}{(a-x)^2}$

(b) $y = \sqrt{1 + (1-x)^2} = (1+(1-x)^2)^{\frac{1}{2}}$

$2\frac{1}{2}$

$U = 1-X \quad Y = (1-U^2)^{\frac{1}{2}}$

$V = 1-U^2$

$Y = V^{\frac{1}{2}}$

$\frac{du}{dx} = -1 \quad \frac{dY}{dU} = \frac{dY}{dV}\frac{dV}{dU}$

$\frac{dY}{dX} = \frac{du}{dx}\frac{dY}{dV}\frac{dV}{dU} = -(1-x)(1-(1-x)^2)^{-\frac{1}{2}}$

(c) $y = (\sqrt{1 + x^2})^3 = (1+X^2)^{\frac{3}{2}}$

3

$U = 1+X^2 \quad Y = U^{\frac{3}{2}}$

$\frac{dY}{dX} = 2X\left[\frac{3}{2}(1+X^2)^{\frac{1}{2}}\right]$

$= 3X(1+X^2)^{\frac{1}{2}}$

product!!

(d) $x^2y + xy^2 = 2x$

X

$\frac{d(x^2Y)}{dX} + \frac{d(XY^2)}{dX} = 2\frac{dX}{dX}$

$\cancel{2X\frac{dY}{dX}} + \cancel{2Y}\frac{dY}{dX} = 2$

$\frac{dY}{dX}(X+Y) = 1 \Rightarrow \frac{dY}{dX} = \frac{1}{X+Y}$

(e) $y = x^2\sqrt{1 + 2x}$

$2\frac{1}{2}$

$\frac{d(1+2X)^{\frac{1}{2}}}{dX}$

$U = 1+2X \quad Y_1 = U^{\frac{1}{2}}$

$\frac{d(1+2X)^{\frac{1}{2}}}{dX} = (1+2X)^{-\frac{1}{2}}$

$\frac{dY}{dX} = X^2(1+2X)^{-\frac{1}{2}} + 2(1+2X)^{\frac{1}{2}}X$ simplify!

2. Integrate (i.e. antidifferentiate) each of the following expressions.

(a) $\frac{dy}{dx} = 5x^3 + 3x^{-2} - 5$

3

$Y = \frac{5}{4}X^4 - 3X^{-1} - 5X + C$

(b) $\frac{ds}{dt} = (5t + 3)^{5/2}$

$U = 5t + 3$

$\frac{du}{dt} = 5$

$2\frac{1}{2}$

$S = \frac{1}{5}\frac{2}{7}(5T+3)^{\frac{7}{2}} = \frac{2}{35}(5T+3)^{\frac{7}{2}} + C$?

(c) $\frac{dw}{dz} = (z^2 + 3)^2 = z^4 + 6z^2 + 9$

3

$W = \frac{1}{5}Z^5 + 2Z^3 + 9Z + C$

(d) $\frac{dy}{dx} = x^2 \sqrt[3]{5x^3 + 9} = X^2(5X^3+9)^{\frac{1}{3}}$

3

$U = 5X^3 + 9 \qquad \frac{du}{dx} = 15X^2$

$Y = \frac{1}{15\cancel{X^2}}\cancel{X^2}\frac{3}{4}(5X^3+9)^{\frac{4}{3}}$

$Y = \frac{1}{20}(5X^3+9)^{\frac{4}{3}} + C$

(e) $\frac{dz}{dx} = \frac{dy}{dx}\frac{d^2y}{dx^2}$

$Z = Y\frac{dY}{dx} + C$

X

3. (a) Using only the definition of the derivative (i.e. method of increments, four step method, etc.) compute the first derivative of

$$y = \frac{1}{x-1}$$

$$Y + \Delta Y = \frac{1}{(X+\Delta X)-1}$$

$$\Delta Y = \frac{1}{(X+\Delta X)-1} - \frac{1}{X-1}$$

$$\Delta Y = \frac{(X-1)-(X+\Delta X-1)}{(X-1)(X+\Delta X-1)}$$

$$\Delta Y = \frac{(\cancel{X}\cancel{-1}-\cancel{X}-\Delta X\cancel{+1})}{(X-1)(X+\Delta X-1)}$$

$$\Delta Y = \frac{-\Delta X}{(X-1)(X+\Delta X-1)}$$

$$\frac{\Delta Y}{\Delta X} = \frac{-1}{(X-1)(X+\Delta X-1)}$$

$$\frac{dY}{dX} = \lim_{\Delta X \to 0} \frac{\Delta Y}{\Delta X} = \frac{-1}{(X-1)^2}$$

5

(b) Let u and v be continuous, differentiable functions of x. Prove that if y = uv that

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$Y = UV$$

$$Y + \Delta Y = (U+\Delta U)(V+\Delta V)$$

$$Y + \Delta Y = UV + U\Delta V + V\Delta U + \Delta U \Delta V$$

$$\Delta Y = U\Delta V + V\Delta U + \Delta U \Delta V$$

$$\frac{\Delta Y}{\Delta X} = U\frac{\Delta V}{\Delta X} + V\frac{\Delta U}{\Delta X} + \Delta U\frac{\Delta Y}{\Delta X}$$

$$\frac{dY}{dX} = U\frac{dV}{dX} + V\frac{dU}{dX} \left(\lim_{\Delta X \to 0} \Delta U\right)\frac{dY}{dX}$$

$$\left(\lim_{\Delta X \to 0} \Delta U\right)\frac{dY}{dX} = 0$$ for as $\Delta X \to 0$, $\Delta U \to 0$

$$\frac{dY}{dX} = U\frac{dV}{dX} + V\frac{dU}{dX}$$

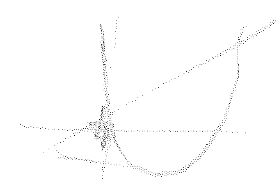
4½

$3Y = 2X + 5$
$3Y = 4X^2 - 2X - 3$
Graphs inter at
$X = 2$ and $X = -1$

$Y = \frac{4X^2}{3} - \frac{2X}{3} - 3$
Integral of $Y = \frac{4}{9}X^3 - \frac{1}{3}X^2 - 3X + C$
$C = 0$
Area
$\frac{4}{9}(8) - \frac{1}{3}(4) - 6 \qquad -\frac{4}{9} - \frac{1}{3} + 3$
$= \frac{32}{9} - \frac{12}{9} - \frac{54}{9} \qquad -\frac{4}{9} - \frac{3}{9} + \frac{27}{9}$
$= \frac{32}{9} - \frac{66}{9} = -\frac{34}{9} \qquad \frac{20}{9}$

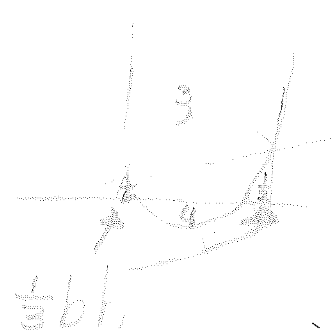
Area inc. par. & X axis $= \frac{36}{9} = 4$

$Y = \frac{2}{3}X + \frac{5}{3}$
Integ of $Y = \frac{1}{3}X^2 + \frac{5}{3}X$
Of $X = 2$
$\frac{4}{3} + \frac{10}{3} = \frac{14}{3}$
$X = -1$
$\frac{1}{3} - \frac{5}{3} = -\frac{4}{3}$
Area under line $= \frac{9}{3} = 3$



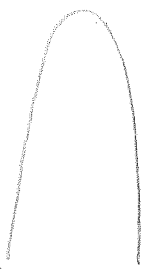
On par. of $Y = 0$, $X = \frac{3}{4}$, or $-\frac{1}{2}$

~~From~~ $\cancel{\frac{1}{2}}$ -1
$\cancel{-\left(\frac{4}{9}\right)\left(\frac{1}{8}\right) - \frac{1}{12} + \frac{3}{2}}$
$\cancel{\frac{4}{72} - \frac{1}{12} + \frac{3}{2}}$ $\frac{1}{3}$
$\frac{4 - 6 + 78}{72} = 1$



Total area $= (3 + 4) - \left(\frac{1}{6} + \frac{3}{6}\right)$
$= 7 - \frac{2}{3} = 6\frac{1}{3}$

4. (a) A ball is thrown upward from the surface of Planet X with an initial velocity of 100 ft/sec. What is the maximum height the ball will reach if the acceleration due to gravitational attraction on Planet X is 25 ft/sec^2?



$V_i = 100$ $\quad V_f = 100$
$t = 0$
$s = 0$

$a = -25$
$V = -25t + C_1$
at $t = 0$, $V = 100 \Rightarrow C_1 = 100$
$V = -25t + 100$
$s = -\frac{25}{2}t^2 + 100t + C_2$
at $s = 0$, $t = 0 \Rightarrow C_2 = 0$

5

$\frac{ds}{dt} = -25t + 100 = 0$
$t = 4$
$\frac{d^2s}{dt^2} = -25 < 0$ so $t = 4$ is a relitive max

$s = -\frac{25}{2}(4)^2 + 100(4)$
$= -200 + 400$
$s_{MAX} = 200$ ft.

(b) The landing speed of an airplane (i.e. the speed at which it touches the ground) is 100 miles/hour. The airplane decelerates at a constant rate and comes to a rest after traveling 1/4 mile along a straight landing strip. Find the deceleration in miles/(hour)^2.

$V_i = 100$
$s_i = 0$
$t_i = 0$
$a = C$ $[c < 0]$

$\longrightarrow$

$V_f = 0$
$s_f = \frac{1}{4}$
$t_f = ?$
$a = C$

5

C deceleration
$a = C$
$V = -Ct + C_2$
at $V = 100$, $t = 0$
$C_2 = 100$
$V = -Ct + 100$
$s = -\frac{C}{2}t^2 + 100t + C_2$
at $s_i = 0$, $t = 0$
$s = -\frac{C}{2}t^2 + 100t = \frac{1}{4}$
$t = 1/200$ hr
if $V = 0$, $t = \frac{1}{200}$ hr
$V = -Ct + 100$
$0 = -\frac{1}{200}C + 100$
$100 = \frac{1}{200}C$
$C = 20{,}000$ m

$Y = \frac{4}{3}x^2 - 2x - 3$

$\cancel{\frac{4}{3}x^2 - 2x}$

$4x^2 - 2x = 3Y + 3$

$2x^2 - x = \frac{3}{2}(Y + 1)$

$x^2 - \frac{1}{2}x = \frac{3}{4}(Y + 1) + \frac{1}{8}$

$\left(x - \frac{1}{4}\right) = \frac{6}{8}Y + \frac{7}{8}$

$\left(x - \frac{1}{4}\right)^2 = \frac{6}{8}\left(Y + \frac{7}{6}\right)$

$Y' = \frac{4}{3}x^2 - \frac{2}{3}x - 3$

$Y = \frac{4}{9}x^3 - \frac{1}{3}x^2 - 3x + C_1$

when $x = \frac{1}{4}$, $Y = 0$

$0 = \left(\frac{4}{9} \cdot \frac{1}{16}\right) - \frac{1}{48} - \frac{3}{4} + C_1$

$C_1 = \frac{1}{144} + \frac{1}{48} + \frac{3}{4}$

$= \frac{1}{144} + \frac{3}{144} + \frac{36}{144} = \frac{38}{144} = \frac{19}{64}$

$Y = \frac{4}{9}(x)^3 - \frac{1}{3}x^2 - 3x + \frac{19}{64}$

from $-\frac{1}{2}$ to 2

$A = \left(\frac{32}{9} - \frac{4}{3} - 6 + \frac{19}{64}\right) + \left(-\frac{4}{72} - \frac{1}{12} + \frac{3}{2} + \frac{19}{64}\right)$

$= \frac{32}{9} - \frac{4}{3} - 6 + \frac{19}{64} - \frac{4}{72} - \frac{1}{12} + \frac{3}{2} + \frac{19}{64}$

$= \frac{[illegible]}{576}$

$64 \times 9 = 576$

1254 - 3872

5. Sketch the curves

$$3y = 2x + 5$$

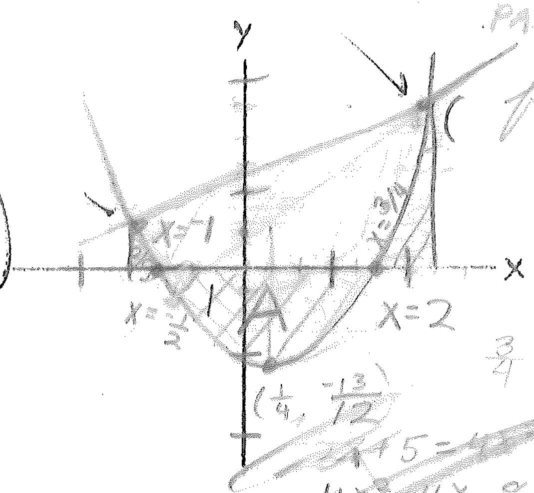
$$3y = 4x^2 - 2x - 3$$

and find the area contained between them.

(SEE BACK OF THIS PAGE 4) for work

I did, but couldn't find it.

1



$3Y = 2X + 5$

$Y = \frac{2}{3}X + \frac{5}{3}$

$X=0,\ Y = \frac{5}{3}$

$Y = \frac{4}{3}X^2 - \frac{2}{3}X - 1$

$X = 0,\ Y = -1$

$\dot{Y} = \frac{8}{3}X - \frac{2}{3} = 0$

$\frac{8}{3}X = \frac{2}{3}$

$X = \frac{1}{4}$

$Y = \frac{4}{3}\left(\frac{1}{4}\right)^2 - \frac{2}{3}\left(\frac{1}{4}\right) - 1$

$= \frac{1}{12} - \frac{2}{12} - \frac{12}{12} = -\frac{13}{12}$

min of par at $\left(\frac{1}{4}, -\frac{13}{12}\right)$

$\frac{3}{4}\ \frac{1}{3}\ \frac{1}{4}$

$2X + 5 = 4X^2 - 2X - 3$

$4X^2 - 4X - 8 = 0$

$X^2 - X - 2 = 0$

$(X-2)(X+1) = 0$

intersect at $X = 2$

and at $X = -1$

$0 = 4X^2 - 2X - 3$

$= (4X-3)(2X+1)$

par. int. X axis at

$X = \frac{3}{4}$ and $X = -\frac{1}{2}$

~~Area cut off~~

Area A

$\dot{Y} = \frac{4}{3}X^2 - \frac{2}{3}X - 1$

$Y = \frac{4}{9}X^3 - \frac{1}{3}X^2 - X + C$

$Y = \frac{4}{9}(X^3) - \frac{1}{3}(X)^2 - X$

$3Y = 2X + 5$

$3Y = 4X^2 - 2X - 3$

$2X + 5 = 4X^2 - 2X - 3$

$4X^2 - 4X - 8 = 0$

$X^2 - X - 2 = 0$

$(X - 2)(X + 1) = 0$

graphs intersect at $X = -1$ and $X = 2$

if $X = 2$, $Y = 3$

if $X = -1$, $Y = 1$

$3Y = 4X^2 - 2X - 3$

$4X^2 - 2X = 3Y + 3$

$\left(X^2 - \frac{1}{2}X + \frac{1}{16}\right) = \frac{3}{4}Y + \frac{3}{4} + \frac{1}{16}$

$\left(X - \frac{1}{4}\right)^2 = \frac{12}{16}Y + \frac{13}{16}$

$\left(X - \frac{1}{4}\right)^2 = \frac{12}{16}\left(Y + \frac{13}{12}\right)$

$P = \frac{6}{16} = \frac{3}{8}$

Vertex $= \left\{\frac{1}{4}, -\frac{13}{12}\right\}$

Translating to $\bar{X}$ axis

$P = \frac{3}{8}$

$X = \frac{1}{4}$ $\quad Y = \frac{11}{12}$

$\left(\bar{X} - \frac{1}{4}\right)^2 = \frac{3}{4}\left(\bar{Y} + \frac{11}{12}\right)$

$\bar{X}^2 - \frac{1}{2}\bar{X} + \frac{1}{8} = \frac{6}{8}\bar{Y} + \frac{11}{16}$

$\frac{3}{4}\bar{Y} = \bar{X}^2 - \frac{1}{2}\bar{X} - \frac{9}{16}$

$\bar{Y} = \frac{4}{3}\bar{X}^2 - \frac{2}{3}\bar{X} - \frac{3}{4}$

Integral $\bar{Y} = \frac{4}{9}\bar{X}^3 - \frac{1}{3}\bar{X}^2 - \frac{3}{4}\bar{X} + C$

$C = 0$

$\therefore$ Area A $= \frac{4}{9}(8) - \frac{1}{3}(4) - \frac{3}{4}(2)$

$= \frac{32}{9} - \frac{4}{3} - \frac{3}{2}$

$= \frac{64 - 24 - 27}{18}$

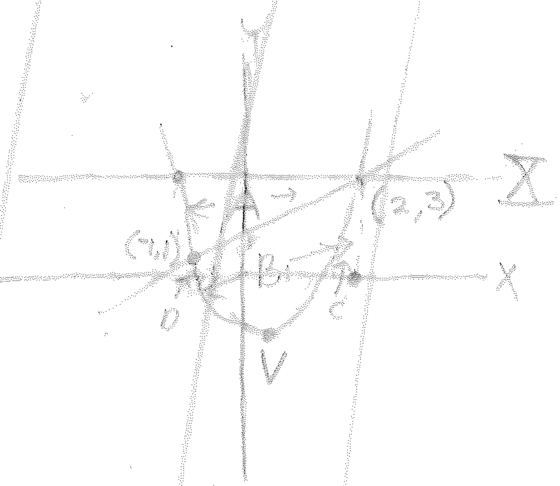
$= \frac{13}{18}$

$Y = \frac{2}{3}X + \frac{5}{3}$

$\bar{Y} = \frac{2}{3}\bar{X} - \frac{1}{3}$

Integral of $\bar{Y} = \frac{1}{3}\bar{X}^2 - \frac{1}{3}\bar{X}$ with $C = 0$

$\frac{4}{3} + \frac{2}{3} = 2$



Area B $= -\frac{4}{9} - \frac{1}{3} + \frac{3}{4}$

$= \frac{-16 - 12 + 27}{36} = -\frac{1}{36}$

Area A + Area B $= \frac{17}{36}$

6. You are given the function $y = \dfrac{x^2}{x - 1}$

(a) What are the range and domain of the function? (Hint: determine range by finding the y values which yield real values of x)

1

domain = all $x \neq 1$
range = $(-\infty, 0) \cup (1, \infty$

(b) For what values of x is the function continuous? For what values does its derivative exist?

X

$\dot{y} = \dfrac{(x-1)2x - x^2}{(x-1)^2} = \dfrac{2x^2 - 2x - x^2}{(x-1)^2} = \dfrac{x^2 - 2x}{(x^2 - 2x + 1)}$

$\ddot{y} = \dfrac{(x-1)^2(2x-2) - (2x-2)(x^2-2x)}{(x-1)^4}$

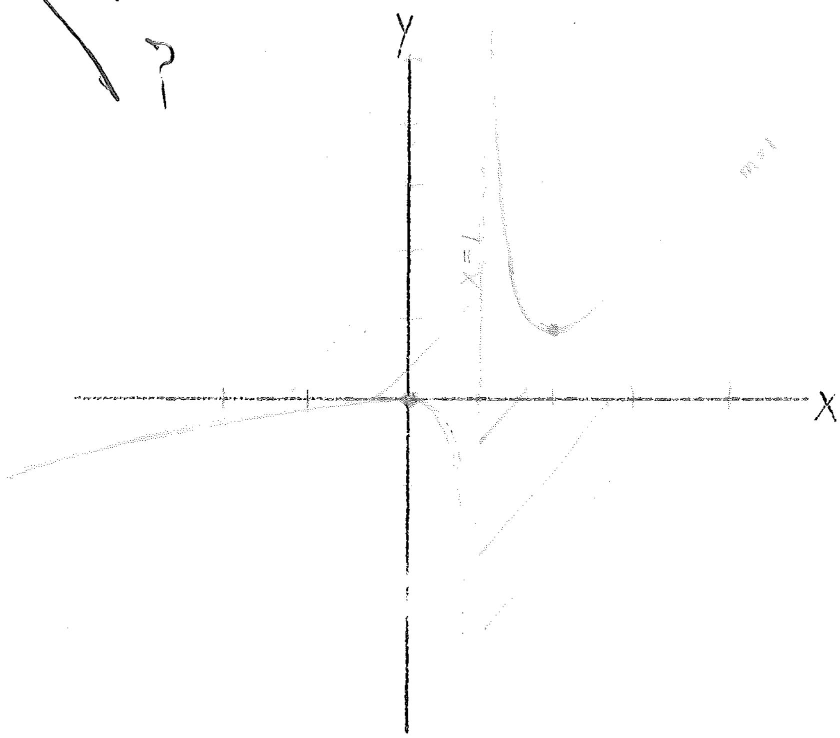
$x = 2$, $y = 1$, $\ddot{y} = 2 > 0$ MIN

$x = 0$, $y = 0$, $\ddot{y} = -2$ MAX

(c) Sketch the function. Be sure to indicate the coordinates of intersections with the coordinate axes and relative maxima and minima if they exist. Also indicate the regions where the curve is concave up and those for which it is concave down.

slope ≠ 1

2 ?



XY int
(0,0)
excluded regions
if $x < 1 \Rightarrow y > 0$
if $x > 1 \Rightarrow y < 0$

7. Consider the curve represented by the function

$$y = x^3 + x$$

(a) What is the smallest value that the slope of this curve can have? At what point on the curve does it occur?

2

$Y = X^3 + X$

$Y' = 3X^2 + 1$

$Y'' = 6X$

min slope at $X = 0$

at pt $(0,0)$

(b) Write the equation(s) of the line(s) tangent to the curve at the point(s) where the slope is equal to 4.

2

$Y' = 3X^2 + 1$

$3X^2 + 1 = 4$

$3X^2 = 3$

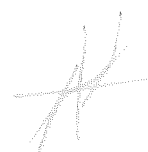
$X^2 = \frac{5}{3}$ 1

$X = \pm\sqrt{\frac{5}{3}}$ ± 1

at $X = +\sqrt{\frac{5}{3}}$ $Y = \left(\frac{5}{3}\right)^{\frac{3}{2}} + \left(\frac{5}{3}\right)^{\frac{1}{2}}$

$Y - \left[\left(\frac{5}{3}\right)^{\frac{3}{2}} + \left(\frac{5}{3}\right)^{\frac{1}{2}}\right] = 4\left[X - \left(\frac{5}{3}\right)^{\frac{1}{2}}\right]$

$Y - \left(\frac{5}{3}\right)^{\frac{3}{2}} - \left(\frac{5}{3}\right)^{\frac{1}{2}} = 4X - 4\left(\frac{5}{3}\right)^{\frac{1}{2}}$



$Y + 3\left(\frac{5}{3}\right)^{\frac{1}{2}} - \frac{5}{3}^{\frac{3}{2}} - 4X = 0$

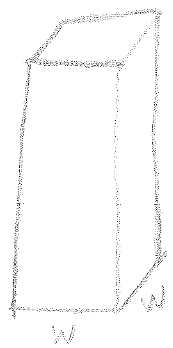
at $X = -\left(\frac{5}{3}\right)^{\frac{1}{2}}$ $Y = -\left(\frac{5}{3}\right)^{\frac{3}{2}} - \left(\frac{5}{3}\right)^{\frac{1}{2}}$

$Y + \left(\frac{5}{3}\right)^{\frac{3}{2}} + \left(\frac{5}{3}\right)^{\frac{1}{2}} = 4X + 4\left(\frac{5}{3}\right)^{\frac{1}{2}}$

$Y + \left(\frac{5}{3}\right)^{\frac{3}{2}} - 3\left(\frac{5}{3}\right)^{\frac{1}{2}} - 4X = 0$

8. A water tank is to have a square base and an open top and hold 1000 gallons. If the base is twice as costly as the sides, what proportions give the minimum material cost? (Hint: to convert gallons to cubic feet multiply by 0.134 cu ft/gallon)

h & w in $(\text{gal})^{\frac{1}{3}}$



$$V = w^2 h \qquad S = 4wh + 2w^2$$

$\Downarrow$ (taking in account area ... of base)

$$1000 = w^2 h$$

$$h = \frac{1000}{w^2}$$

1000 (gal)

$$S = 4\cancel{w}\left(\frac{1000}{w^{\cancel{2}}}\right) + 2w^2$$

$$= 2w^2 + 4000w^{-1}$$

$$\frac{ds}{dw} = 4w - 4000w^{-2} = 0$$

$$\frac{d^2s}{dw^2} = 4 + 8000w^{-3}$$

$$4w = 4000w^{-2}$$

$$w^3 = 1000$$

$$w = 10\,\text{gal}^{\frac{1}{3}} \qquad V = 1000 = w^2 h$$

$$h = 10\,\text{gal}^{\frac{1}{3}}$$

4

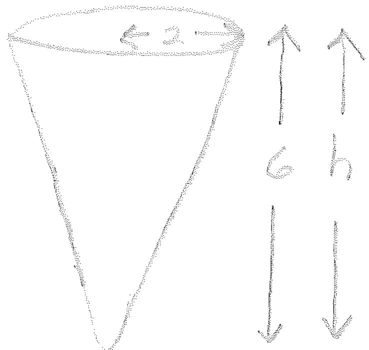
$$w = h = \frac{(.134)^{\frac{1}{3}}\ \text{ft}}{\text{gal}^{\frac{1}{3}}} \times 10\,\text{gal}^{\frac{1}{3}}$$

$$10 = (1000)^{\frac{1}{3}}$$

$$h = w = (134)^{\frac{1}{3}}\ \text{ft}$$

max or min only?

9. A ~~conical paper~~ cup of radius 2 inches and height 6 inches is leaking water at the rate of one cubic inch/minute. At what rate is the level of the water being lowered when the height of the water is one ~~inch.~~



$\frac{dV}{dt} = -1$ $\quad \frac{dh}{dV}$

$V = \frac{1}{3}\pi r^2 h$

$r = 2$ only at one time

$V = \frac{4}{3}\pi h$

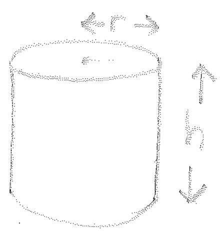
$h = \frac{3}{4}\frac{V}{\pi}$

$\frac{dh}{dV} = \frac{3}{4\pi}$ $\quad \frac{dh}{dV} = \frac{3}{4\pi}\frac{dV}{dt}$

$\frac{dh}{dt} = \frac{3}{4\pi}$ inch/min

(1)

(b) The height of a circular cylinder is being increased at the rate of 4 inches/min. If the volume of the cylinder remains constant, at what rate must the radius be decreasing?



$\frac{dh}{dt} = 4$ $\quad \frac{dr}{dt} = \frac{dh}{dt}\frac{dr}{dh}$

$V = \pi r^2 h$ $\quad S = 2\pi r^2 + 2\pi r h$

$r^2 = \frac{V}{\pi h}$

$r = \left(\frac{V}{\pi} h^{-1}\right)^{\frac{1}{2}} > 0$

$= \left(\frac{V}{\pi}\right)^{\frac{1}{2}} (h)^{-\frac{1}{2}}$

$\frac{dr}{dh} = -\frac{1}{2}\left(\frac{V}{\pi}\right)^{\frac{1}{2}} h^{-\frac{3}{2}}$ $\frac{dh}{dt}$?

$\frac{dr}{dt} = \frac{-\cancel{1}\sqrt{\frac{V}{\pi}} \cdot 4}{2 h^{3/2}} = \frac{-2\sqrt{r^2 h}}{h^{3/2}} = \frac{-2r}{h}$

(4)

10. (a) Using the fact that

$$\lim_{\theta \to 0} \left(\frac{\sin \theta}{\theta}\right) = 1,$$

Determine $\lim_{x \to 0} \left(\frac{\sin 5x}{\sin 3x}\right)$. (Hint: First express $\lim_{x \to 0} \left(\frac{\sin 5x}{\sin 3x}\right)$ as a product of quotient of two limits.

$$\lim_{X \to 0} \left(\frac{\sin 5X}{\sin 3X}\right) = \frac{5 \sin 3X \cos 5X - 3 \sin 5X \cos 3X}{\sin^2 3X}$$

NO.

$$= \frac{5 \cancel{\sin 3X} \cos 5X}{\sin^{\cancel{2}} 3X} - \frac{3 \sin 5X \cos 3X}{\sin^2 3X}$$

$$= \frac{5 \cos 5X}{\sin 3X} - \frac{3 \sin 5X \cot 3X}{\sin 3X}$$

$$= \frac{5 \cos 5X}{\sin 3X} - \frac{3 \sin 5X}{\sin 3X \tan 3X}$$

(b) Find $\frac{dy}{dx}$ when

1) $y = x^2 \sin 3x$

(2)

$$= x^2 3 \cos 3X + (\sin 3X) 2X$$

$$= 3X^2 \cos 3X + 2X \sin 3X$$

2) $y = \sqrt{2 + \cos x} = (2 + \cos x)^{\frac{1}{2}}$

$U = 2 + \cos X \qquad Y = U^{\frac{1}{2}}$

(2)

$$\frac{dY}{dX} = \frac{1}{2} \sin X (2 + \cos X)^{-\frac{1}{2}}$$

3) $y = 3\cos^2 2x - 3\sin^2 2x$

$$\frac{dY}{dX} = 12 \sin^2 2X - 12 \cos^2 2X$$

Bob Marks

C

1. This exam consists of seventeen (17) questions for a total of 75 points. The nature of the questions are short essays, fill-in-the-blank, and multiple choice. The weight of each question is indicated at the end of the question.

2. Attached to the end of the exam are questions for extra credit of ten (10) points. You need not answer these questions if you desire not to.

3. You do not have to answer all seventeen questions before answering the extra-credit problems. In other words, you can leave several of the seventeen questions blank and make up for them by answering the extra-credit questions.

4. The Honor System applies for this examination.

$\sim -12 + 3 = -9$

36

CIVIL WAR

Although the basic cause of the American Civil War is often accepted as centering on the conflict that existed between the North and South over the right of secession, other reasons can be cited as adequate causes for the War.

1. Briefly explain the following: (4 points)

a. the relationship of the admission of new states (as a slave or free state) to Congress. The admission of new states gives more congressmen to Congress; thus a free state would give more "northern" votes and [illegible] "southern" votes in legislation dealing with conflicting N-S ideals

b. the consequent impact that this relationship had on the economic systems of the North and South.

The passage of the Mo. Compromise, to keep free-slave votes at a balance

-1

2. Briefly explain how shouts for democracy and liberty could have been strong ideological reasons for the start of the Civil War. (4 points)

The morality of slavery was questioned, thus the democracy & liberty of the slave could have been a reason for the start of the Civil War.

At the outset of the Civil War, the armies of the North and South differed greatly in quality and quantity.

3. A majority of the Regular Army officers joined the Army of the SOUTH while most of the enlisted personnel went to the side of the NORTH. (North or South) (2 points)

4. [illegible] of spreading [illegible] out among the [illegible] WHY? a, the spreading out of the regular army would have mixed experienced men with un-experienced men, and the latter could of learned from the former, ~~the~~ resulting in a better army

The [illegible] available to any army unquestionably has a great influence upon the over-all strategy which that army will develope in an attempt to defeat an enemy. This was no exeption in the case of the Northern strategy which was directed against the resources of the South. Their strategy (North) involved three basic actions or movements. These were: (3 points) Select best answer:

5. a. Control the Ohio River, close the ports on the East coast to European trade, move on to Charleston.

b. Control the Wabash River, close the Southern ports, move on to Richmond.

c. Control the Mississippi River thus dividing the South, close the southern ports, and move on to Charleston.

d. Close the southern ports, control the Mississippi River thus dividing the South, move on to Richmond.

6. What should have been the strategy of the South and Why? What strategy did Davis adopt? (3 points)

Davis moved to offensive too quickly, thus thinning his already troops. He should of stayed on the defense, fought the North in the South thus wearing the north

WORLD WAR I

The German plan for World War I, commonly known as the Schlieffen Plan, [illegible] a thrust into France first, followed by an advance into Russia

7. What two assumptions (relative to France and Russia) was this decision based upon? (2 points)

1) ~~Hans on the outskirts of Paris would help in~~ crushing Paris would crush France

2) Russia would not be able to mobilize

-1

[illegible]

His greatest mass should have been on the ~~inner~~ outside of the swing around Paris, but he took away some troops (too many) to help elsewhere.

b. [illegible]: His greatest mass should have been on the outside of the swing around Paris, but he took out too many troops to help elsewhere.

−1

c. The plan that the French adopted, known as [illegible] [illegible] was a [illegible] [illegible]. What was this plan? 2 PTS

To attack over [illegible] Lorraine. The Germans [illegible]

On the extreme right wing of the forces that were attacking [illegible] France according to the Schlieffen Plan was the First German Army [illegible] by General Von Kluck. As the First Army neared Paris, Kluck decided to assist the commander of the Second Army, General Von Bulow, who was being [illegible] by the French east of Paris. Hence Kluck moved his First Army Southeast of Paris, a movement which sealed the doom of the Schlieffen Plan and the German hope of dealing a lightning swift blow to the French. Why? (2 [illegible])

10. Explanation:

It drained the mass of the Russian troops swinging around Paris; they [illegible] to fast. Also, by [illegible] the French were mobilizing an army where the [illegible] were [illegible] to [illegible]

−2

-1

Worker (Proletariats) vs. ~~Ruling class~~ (Bourgeoisie) (OR SYSTEM)

-1 The ruling [illegible] bring [illegible] of the workers and rules them economically [illegible]

Russia was [illegible] for Democracy, Unrest [illegible] by various parties.

-2 ~~[illegible]~~ Russia [illegible] was [illegible]

What [illegible]

[illegible] (2 points)

(MORE THAN ONE ANS)

1PT 14. [illegible] normally has two political parties to provide for control of the masses.

b. terroristic police force

[illegible] single leader head of <u>at least</u> two parties in order to appease the masses.

d. communications monopoly

e. driven by an ideology

f. control of economy is highly localized.

g. weapons monopoly (unlimited production of weapons)

-1/3

15. However, the differences between a Fascist state and a Communist state are quite pronounced. Briefly discuss the differences in the type of struggle that the Fascist Revolution and the Communist Revolution represented (i.e., opponents of the respective revolutions) (2 points) STRUGGLE

The communist revolution was a class revolution while the Fascist revolution was nationalistic.

16. Briefly discuss the permanency of the state under Fascism and Communism. (2 points)

The Fascist state is supposedly permanent while the Communist state fades (in theory)

<u>HITLER'S GERMANY</u>

As matters stood in July 1919, Germany and Austria were prostrate, Hungary, and fumbling for life. The Allies had continued to humiliate Germany and the Communist party was growing. Thus was the situation when Adolf Hitler entered and provided, in his National Socialist Party (NAZI), a rallying point for Germans striving to free themselves both from the Allied Yoke and from Communism. By March 1933, the Nazi dictatorship was firmly established. <u>Military Heritage of America</u> states that for the next six years,

-1/2

To keep [illegible] out of [illegible]
and [illegible] country.
To split Poland

-2

Hitler pronounced
the Germans as the master
race, thus starting nationalism,
and giving reason [illegible] taking
over [illegible] countries. He
pronounced Jews as the inferior race
(& so [illegible] them almost
out)

Being the [illegible] race [illegible] Germany
-1/2 must have living space & must
be [illegible] [illegible], [illegible]
reason for going to war [illegible]

therefore, [illegible] need what?

EXTRA CREDIT

1. The following questions are from Mr. Kraft's lecture on the Second World War: (2 points)

+1

2) Political Stability
Economic Stability
[illegible]

2. Explain the following statement with respect to the results of the treaty between the Allies and Germany following WW I:

"For the first time in human history, the scourge of war came to be regarded by a large part of mankind as a primary evil." (3 points)

The allies were rationalizing their split of Germany & damage in the [illegible]

3. The basic principle of the League of Nations was the concept of "collective security". What was this principle? (2 points)

Of all countries in settlement and have a common body of communication in their countries through the League, frictions could be settled peacefully

+1

4. Between WW I and WW II Britain built up a doctrine of air power based on what was called "strategic" bombardment. Explain this strategy. (2 points)

5. What land or territory did the Munich Agreement of 1938 provide to Hitler? (1 point)

Sudetenland from Czech.

+1

11. Example 8. *An application to chemistry*

Suppose that, in a certain chemical reaction, materials α and β unite in the ratio of 2 grams to 3 grams, and form material γ. Suppose that we start with 8 grams of α and 9 grams of β and that, under the conditions of this experiment, the rate at which γ is formed at time t is proportional to the product of the amounts of α and β present at time t. Now if x grams of γ have already been formed at time t, then $\frac{2}{5}x$ grams of α and $\frac{3}{5}x$ grams of β must have been used up, and there remain $8 - \frac{2}{5}x$ grams of α and $9 - \frac{3}{5}x$ grams of β. Thus

$$\frac{dx}{dt} = C\left(8 - \frac{2}{5}x\right)\left(9 - \frac{3}{5}x\right) = K(20 - x)(15 - x), \tag{9}$$

where C and $K = 6C/25$ are constants determined by chemical experiment. We can rewrite Eq. (9) as

$$\frac{1}{(20 - x)(15 - x)}\frac{dx}{dt} = K,$$

and then, by our partial fraction technique, as

$$\left(-\frac{1}{5}\frac{1}{20 - x} + \frac{1}{5}\frac{1}{15 - x}\right)\frac{dx}{dt} = K.$$

Integrating with respect to t, we get

$$\int -\frac{1}{20 - x}\frac{dx}{dt}\,dt + \int \frac{1}{15 - x}\frac{dx}{dt}\,dt = \int 5K\,dt$$

or

$$\log(20 - x) - \log(15 - x) = 5Kt + C', \tag{10}$$

where C' is a constant of integration.

But at time $t = 0$, no γ was yet present; $x = 0$. Hence in Eq. (10),

$$\log 20 - \log 15 = C' \qquad \text{or} \qquad C' = \log \tfrac{20}{15} = \log \tfrac{4}{3},$$

and Eq. (10) reads

$$\log(20 - x) - \log(15 - x) = 5Kt + \log \tfrac{4}{3},$$

$$\log \frac{3(20 - x)}{4(15 - x)} = 5Kt,$$

$$\frac{3(20 - x)}{4(15 - x)} = e^{5Kt}.$$

At any time t there are

$$x = 15\,\frac{e^{5Kt} - 1}{e^{5Kt} - .75} \tag{11}$$

grams of γ. As a common-sense check, 6 grams of α will ultimately combine with the 9 grams of β originally present to give 15 grams of γ; note that Eq. (11) says that $x \to 15$ as t grows beyond all bounds.

$$4.\ \int \frac{7x+6}{x^3+x^2-6x}\,dx = \int \frac{7x+6}{x(x^2+x-6)}\,dx = \int \frac{7x+6}{x(x+3)(x-2)}\,dx$$

$$= \int \left[\frac{\overset{A}{-1}}{x} + \frac{\overset{B}{-1}}{x+3} + \frac{\overset{C}{2}}{x-2} \right] dx = -\log x - \log(x+3) + 2\log(x-2)$$

$$= \log \frac{(x-2)^2}{x(x+3)} + C$$

$$A(x+3)(x-2) + Bx(x-2) + Cx(x+3) \equiv 7x+6$$

$x=0$: $-6A = 6$, $A=-1$

$x=-3$: $15B = -15$; $B=-1$ or

$x=2$: $10C = 20$, $C=2$

x^2: $A+B+C = 0$

x: $A-2B+3C = 7$

x^0: $-6A = 6$, $A=-1$

$\left.\begin{matrix} B+C=1 \\ -2B+3C=8 \end{matrix}\right\}$ $\begin{matrix} 5C=10 \\ C=2 \\ B=-1 \end{matrix}$

$$5.\ (a)\ \lim_{x\to\frac{\pi}{2}} (\pi-2x)\tan x \qquad 0\cdot\infty$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\pi-2x}{\cot x}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{-2}{-\csc^2 x}$$

$$= 2\sin^2\frac{\pi}{2} = 2$$

$$(b)\ \lim_{x\to\frac{\pi}{2}} \sec 3x \cos 5x \qquad \infty\cdot 0$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\cos 5x}{\cos 3x}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{-5\sin 5x}{-3\sin 3x} = -\frac{5}{3}$$

$$6.\ \int 27x^2 e^{3x}\,dx = 27\left[(x^2)\left(\frac{e^{3x}}{3}\right) - (2x)\left(\frac{e^{3x}}{9}\right) + 2\left(\frac{e^{3x}}{27}\right)\right] + C$$

$$\int u\ v\ dx = u\ v_1 - u'\ v_{11} + u''\ v_{111}$$

$$= e^{3x}(9x^2-6x+2) + C$$

or $u = x^2$ $\quad dv = 27e^{3x}dx$ $\qquad I = 9x^2e^{3x} - 18\int xe^{3x}dx$

$du = 2x\,dx$ $\quad v = 9e^{3x}$

$18\int xe^{3x}dx = 6xe^{3x} - \int 6e^{3x}dx = 6xe^{3x} - 2e^{3x}$

$u = x$ $\quad dv = 18e^{3x}dx$ $\qquad I = (9x^2-6x+2)e^{3x} + C$

$du = dx$ $\quad v = 6e^{3x}$

Key

1. $y = \dfrac{5(x+1)^2(x+2)}{(x+3)(x+4)^2}$ Find values of y and y' when x

$$y = \frac{5(3^2)(4)}{5(6^2)} = \frac{5\cdot 9\cdot 4}{5\cdot 36} = 1$$

$$\log y = \log 5 + 2\log(x+1) + \log(x+2) - \log(x+3) - 2\log(x+4)$$

$$\frac{y'}{y} = \frac{2}{x+1} + \frac{1}{x+2} - \frac{1}{x+3} - \frac{2}{x+4}$$

$$x=2,\ y' = \frac{2}{3} + \frac{1}{4} - \frac{1}{5} - \frac{1}{3} = \frac{1}{3} + \frac{1}{20} = \frac{23}{60}$$

2. $\int (\tan x + \cot x)^2\,dx = \int (\tan^2 x + 2 + \cot^2 x)\,dx$

$$= \int (\sec^2 x + \csc^2 x)\,dx = \tan x - \cot x + C$$

or $\int \left(\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\right)^2 dx = \int \left(\frac{1}{\sin x \cos x}\right)^2 dx = \int \left(\frac{2}{\sin 2x}\right)^2 dx = 4\int \csc^2 2x\,dx$

$$= C - 2\cot 2x$$

Note that $-2\cot 2x = \dfrac{-2}{\tan 2x} = \dfrac{-2}{\frac{2\tan x}{1-\tan^2 x}} = \dfrac{\tan^2 x - 1}{\tan x} = \tan x - \cot x$

3. Find $\int \sec x\,dx$ by first showing $\sec x \equiv \frac{1}{2}\cos x\left[\frac{1}{1+\sin x} + \dots\right.$

$$\frac{1}{2}\cos x\,\frac{1-\sin x + 1 + \sin x}{1-\sin^2 x} = \frac{\cos x}{\cos^2 x} = \frac{1}{\cos x} = \sec x$$

$$\int \sec x\,dx = \frac{1}{2}\int \left(\frac{\cos x}{1+\sin x} + \frac{\cos x}{1-\sin x}\right)dx = \frac{1}{2}\left[\log(1+\sin x) - \log(1-\sin x)\right]$$

$$= \frac{1}{2}\log\frac{1+\sin x}{1-\sin x} + C = \log\frac{1+\sin x}{\cos x} + C = \log\frac{\cos x}{1-\sin x} + C$$

$$= \log\tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + C \quad = \log(\sec x + \tan x) + C$$

since $\sqrt{\dfrac{1+\sin x}{1-\sin x}} = \sqrt{\dfrac{1-\cos(\frac{\pi}{2}+x)}{1+\cos(\frac{\pi}{2}+x)}} = \dfrac{\sin\frac{1}{2}(\frac{\pi}{2}+x)}{\cos(\frac{\pi}{4}+\frac{x}{2})} = \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$

Due Mon Mar 10 & Wed Mar 12 (10 each day)

Page #

Page # 406 399
16 3

$Y = \frac{4}{3}x^2 - 2x - 3$

$\cancel{\frac{4}{3}x^2 - 2x}$

$\cancel{4x^2 - 2x = 3Y + 3}$

$\cancel{2x^2 - x = \frac{3}{2}(Y+1)}$

$\cancel{x^2 - \frac{1}{2}x = \frac{3}{4}(Y+1) + \frac{1}{8}}$

$\cancel{\left(x - \frac{1}{4}\right) = \frac{6}{8}Y + \frac{7}{8}}$

$\cancel{\left(x - \frac{1}{4}\right)^2 = \frac{6}{8}\left(Y + \frac{7}{6}\right)}$

$Y = \frac{4}{3}x^2 - \frac{2}{3}x - 3$

$Y = \frac{4}{9}x^3 - \frac{1}{3}x^2 - 3x + C_1$

when $x = \frac{1}{4}$, $Y = 0$

$0 = \left(\frac{4}{9} \cdot \frac{1}{16}\right) - \frac{1}{48} - \frac{3}{4} + C_1$

$C_1 = \frac{1}{144} + \frac{1}{48} + \frac{3}{4}$

$= \frac{1}{144} + \frac{3}{144} + \frac{36}{144} = \frac{38}{144} = \frac{19}{64}$

$Y = \frac{4}{9}(x)^3 - \frac{1}{3}x^2 - 3x + \frac{19}{64}$

from $\frac{1}{3}$ to 2

$A = \left(\frac{32}{9} - \frac{4}{3} - 6 + \frac{19}{64}\right) + \left(\frac{-4}{72} - \frac{1}{12} + \frac{3}{2} + \frac{19}{64}\right)$

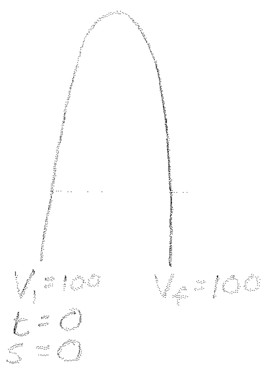
$= \frac{32}{9} - \frac{4}{3} - 6 + \frac{19}{64} - \frac{4}{72} - \frac{1}{12} + \frac{3}{2} + \frac{19}{64}$

$= \frac{\cancel{XXXX} - \cancel{XXX} - \cancel{3456} + \cancel{XXXX} - \cancel{8X} - \cancel{48X} + \cancel{XXXX} + \cancel{XXXX}}{576}$

1254 − 3872

$\begin{array}{r} 64 \\ 9 \\ \hline 576 \end{array}$

4. (a) A ball is thrown upward from the surface of Planet X with an initial velocity of 100 ft/sec. What is the maximum height the ball will reach if the acceleration due to gravitational attraction on Planet X is 25 ft/sec^2?



$a = -25$

$V = -25t + C_1$

at $t = 0$, $V = 100 \Rightarrow C_1 = 100$

$V = -25t + 100$

$s = -\frac{25}{2}t^2 + 100t + C_2$

at $s = 0$, $t = 0 \Rightarrow C_2 = 0$

5

$\frac{ds}{dt} = -25t + 100 = 0$

$t = 4$

$\frac{d^2s}{dt^2} = -25 < 0$ so $t = 4$ is a relative max

$s = -\frac{25}{2}(4)^2 + 100(4)$

$= -200 + 400$

$s_{MAX} = 200$ ft.

(b) The landing speed of an airplane (i.e. the speed at which it touches the ground) is 100 miles/hour. The airplane decelerates at a constant rate and comes to a rest after traveling 1/4 mile along a straight landing strip. Find the deceleration in miles/(hour)^2.

$V_1 = 100$
$s_1 = 0$
$t_1 = 0$
$a = C$ $[c < 0]$

$V_f = 0$
$s_f = \frac{1}{4}$
$t_f = ?$
$a = C$

5

deceleration
$a = -C$

$V = -Ct + C_2$

at $V = 100$, $t = 0$

$C_2 = 100$

$V = -Ct + 100$

$s = -\frac{C}{2}t^2 + 100t + C_2$

at $s_1 = 0$, $t = 0$

$s = -\frac{C}{2}t^2 + 100t = \frac{1}{4}$

$t = 1/200$ hr

if $V = 0$, $t = \frac{1}{200}$ hr

$V = -Ct + 100$

$0 = -\frac{1}{200}C + 100$

$100 = \frac{1}{200}C$

$C = 20,000 \frac{mi}{hr^2}$

$3Y = 2X + 5$

$3Y = 4X^2 - 2X - 3$

$2X + 5 = 4X^2 - 2X - 3$

$4X^2 - 4X - 8 = 0$

$X^2 - X - 2 = 0$

$(X - 2)(X + 1) = 0$

Graph the integral

at $X = -1$ and $X = 2$

at $X = 2,\ Y = 3$

at $X = -1,\ Y = 1$

$3Y = 4X^2 - 2X - 3$

$4X^2 - 2X = 3Y + 3$

$X^2 - \frac{1}{2}X + \frac{1}{16} = \frac{3}{4}Y + \frac{3}{4} + \frac{1}{16}$

$(X - \frac{1}{4})^2 = \frac{2}{6}Y + \frac{13}{16}$

$(X - \frac{1}{4})^2 = \frac{12}{16}(Y + \frac{13}{12})$

$p = \frac{6}{16} = \frac{3}{8}$

V vertex $\{\frac{1}{4}, -\frac{13}{12}\}$

Translating X axis

$p = \frac{3}{8}$

$X = \frac{1}{4} \quad Y = \frac{11}{12}$

$(X - \frac{1}{4})^2 = \frac{3}{4}(Y + \frac{11}{12})$

$X^2 - \frac{1}{2}X + \frac{1}{8} = \frac{6}{8}Y + \frac{11}{16}$

$\frac{3}{4}Y = X^2 - \frac{1}{2}X - \frac{9}{16}$

$Y = \frac{4}{3}X^2 - \frac{2}{3}X - \frac{3}{4}$

Integral $Y = \frac{4}{9}X^3 - \frac{1}{3}X^2 - \frac{3}{4}X + C$

$C = 0$

Area A $= \frac{4}{9}(8) - \frac{1}{3}(4) - \frac{3}{4}(2)$

$= \frac{32}{9} - \frac{4}{3} - \frac{3}{2}$

$= \frac{64 - 24 - 27}{18}$

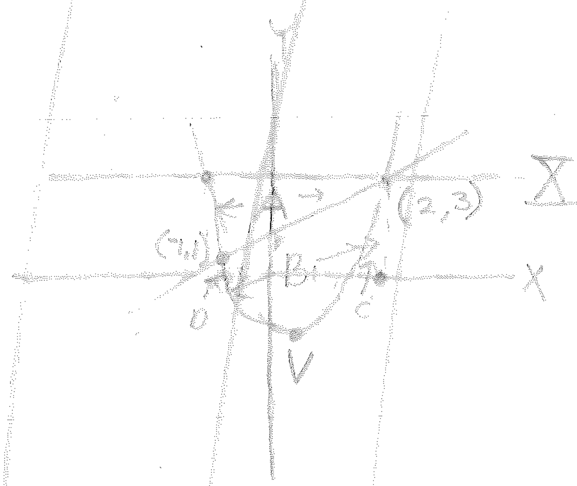
$= \frac{13}{18}$

$Y = \frac{2}{3}X + \frac{5}{3}$

$Y = \frac{2}{3}X - \frac{1}{3}$

Integral of $Y = \frac{1}{3}X^2 - \frac{1}{3}X$ with c = ...

$\frac{4}{3} + \frac{2}{3} = 2$



Area B $= -\frac{4}{9} - \frac{1}{3} + \frac{3}{4}$

$= \frac{-16 - 12 + 27}{36} = \frac{1}{36}$

Area A + ... = ?

5. Sketch the curves

$$3y = 2x + 5$$

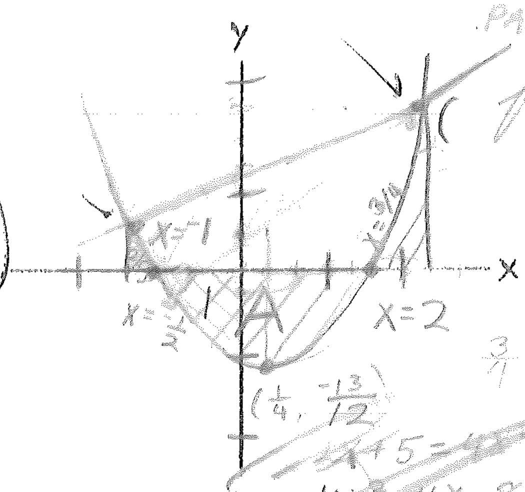
$$3y = 4x^2 - 2x - 3$$

and find the area contained between them.

(SEE BACK OF THIS PAGE 4) for work

I did, but couldn't find it.

1



$3Y = 2X + 5$

$Y = \frac{2}{3}X + \frac{5}{3}$

$X = 0,\ Y = \frac{5}{3}$

$Y = \frac{4}{3}X^2 - \frac{2}{3}X - 1$

$X = 0,\ Y = -1$

$\dot{Y} = \frac{8}{3}X - \frac{2}{3} = 0$

$\frac{8}{3}X = \frac{2}{3}$

$X = \frac{1}{4}$

$Y = \frac{4}{3}\left(\frac{1}{4}\right)^2 - \frac{2}{3}\left(\frac{1}{4}\right) - 1$

$= \frac{1}{12} - \frac{2}{12} - \frac{12}{12} = -\frac{13}{12}$

min of par at $\left(\frac{1}{4}, -\frac{13}{12}\right)$

$2X + 5 = 4X^2 - 2X - 3$

$4X^2 - 4X - 8 = 0$

$X^2 - X - 2 = 0$

$(X-2)(X+1) = 0$

intersect at $X = 2$ and at $X = -1$

$0 = 4X^2 - 2X - 3$

$= (4X - 3)(2X + 1)$

par. int. X axis at $X = \frac{3}{4}$ and $X = -\frac{1}{2}$

~~Area cut off~~

Area A

$Y = \frac{4}{3}X^2 - \frac{2}{3}X - 1$

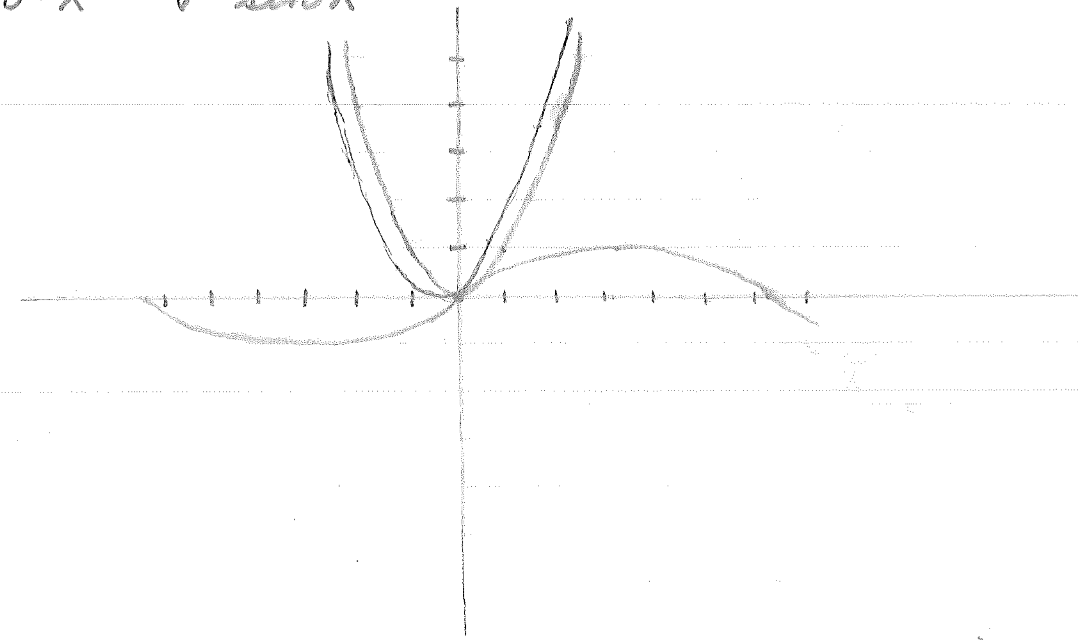
$Y = \frac{4}{9}X^3 - \frac{1}{3}X^2 - X + C$

$Y = \frac{4}{9}(X^3) - \frac{1}{3}(X)^2 - X$

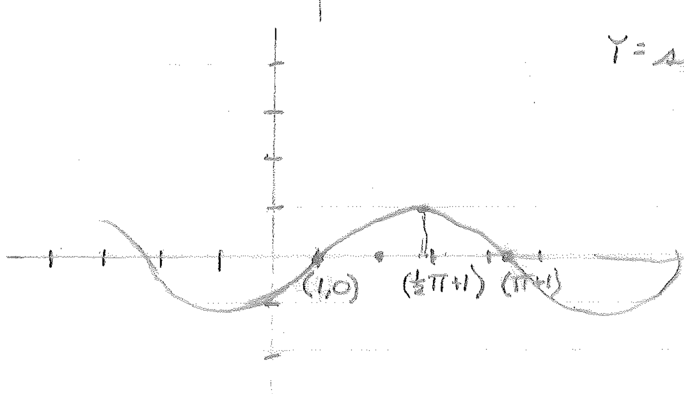
CALC II

5) $Y = X^2 + \sin X$

$U = X^2 \quad V = \sin X$



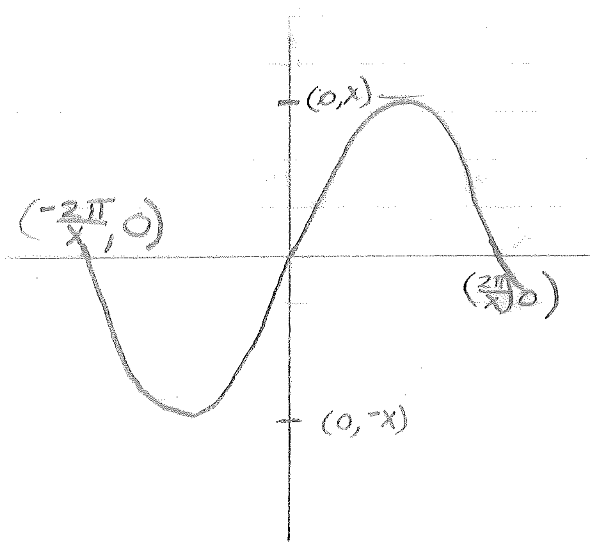
1) g)



$Y = \sin(x-1)$

X	Y
1	0
$\pi+1$	0
$\frac{1}{2}\pi+1$	

3)



Pg 235

1) $\frac{1-\cos x}{x} \cdot \frac{1+\cos x}{1+\cos x} = \frac{1-\cos^2 x}{x(1+\cos x)}$

$\frac{\sin x}{x}\left(\frac{\sin x}{1+\cos x}\right)$

$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x \to 0} \frac{\sin x}{1+\cos x} = 0$

$\lim_{x \to 0} \frac{\sin x}{x}\left(\frac{\sin x}{1+\cos x}\right) = 0$

2) a) $\lim_{x \to 0} \frac{\sin 2x}{x} = \lim_{x \to 0} \frac{2\cos x \sin x}{x}$

$= \lim_{x \to 0} 2\cos x = 2$

c) $\lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} = 1$

e) $\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \cos x \frac{\sin x}{x}$

$= \lim \cos x = 1$

Pg 242

$\frac{\Delta y}{\Delta x} = \frac{\sin(x_0 + \Delta x) - \sin x_0}{\Delta x}$

$= \frac{2\cos \frac{2x+\Delta x}{2} \sin \frac{\Delta x}{2}}{\Delta x}$

$= \frac{2\cos\left(x + \frac{\Delta x}{2}\right) \sin \frac{\Delta x}{2}}{\Delta x}$

$= \cos\left(x + \frac{\Delta x}{2}\right)\left[\frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}}\right]$

$\frac{dy}{dx} = \cos x$

2) $Y = \cos X$

$= \sin\left(\frac{\pi}{2} - X\right)$

$\dot{Y} = {}^{-}\cos\left(\frac{\pi}{2} - X\right)$

$= {}^{-}\cos\frac{\pi}{2}\cos X - \sin\frac{\pi}{2}\sin X$

$= {}^{-}\sin X$

3) $Y = \tan X$

$Y = \frac{\sin X}{\cos X}$

$\dot{Y} = \frac{\cos^2 X + \sin^2 X}{\cos^2 X}$

$= 1 + \frac{\sin^2}{\cos^2}$

$= 1 + \tan^2$

$= \sec^2$

4) $Y = \cot X$

$Y = \frac{\cos X}{\sin X}$

$\dot{Y} = \frac{{}^{-}\sin^2 X - \cos^2 X}{\sin^2 X}$

$= {}^{-}1 - \cot^2$

$= {}^{-}\csc^2$

5) $Y = \sec X$

$= \frac{1}{\cos X}$

$\dot{Y} = \frac{{}^{-}\sin X}{\cos X}\,\frac{1}{\cos X}$

$= {}^{-}\tan X \sec X$

6) $Y = \csc X$

$Y = \frac{1}{\sin X}$

$\dot{Y} = \frac{{}^{-}\cos X}{\sin X^2}$

$= {}^{-}\csc X \cot X$

7) a) $Y = \sin 2X$

$Y = \sin U \quad U = 2X$

$\dot{Y} = 2\cos U$

c) $Y = 3\cos 2X$

$U = 2X \quad Y = 3\cos U$

$\dot{Y} = 2^{-}(3\sin 2X)$

$= -6\sin 2X$

~~k)~~ $Y = (1+\tan^2 X)^{\frac{1}{2}}$

$U = 1+\tan^2 X \quad Y = U^{\frac{1}{2}}$

$\dot{Y} = \tan^2 X \sec X^4 (1+\tan^2 X)^{-\frac{1}{2}}$

~~k)~~ $Y = (1+\tan^2 X)^{\frac{1}{2}}$

$U = 1+\tan^2 X \quad Y = U^{\frac{1}{2}}$

$\dot{Y} = 2\tan X \sec^2 X \frac{1}{2}(1+\tan^2 X)^{-\frac{1}{2}}$

$\dot{Y} = \dfrac{\tan X \sec^2 X}{(1+\tan^2 X)^{\frac{1}{2}}} = \tan X$

k) $Y = (1+\tan^2 X)^{\frac{1}{2}}$

$= (\sec)$

$\dot{Y} = \sec X \tan X$

l) $Y = (2-\cos^2 X)^{\frac{1}{2}}$

$U = 2-\cos^2 X \quad Y = U^{\frac{1}{2}}$

$\dot{Y} = \cancel{1}(2\sin X \cos X)\frac{1}{2}(2-\cos^2 X)^{-\frac{1}{2}}$

$\dot{Y} = \dfrac{\sin X \cos X}{\sqrt{2-\cos^2 X}}$

8) a) $Y = \sin^3 X$

$Y = \sin^2 X \sin X$

$\dot{Y} = \sin^2 X \cos X + \sin^2 X \sin X$

$\dot{Y} = \sin^2 X \cos X + (2\sin X \cos X)\sin X$

$= 3\sin^2 X \cos X$

i) $Y = x\cos\frac{1}{x}$

$U = x^{-1}$ $\qquad Y = \frac{1}{U}\cos U$

$\frac{dU}{dx} = -\frac{1}{x^2}$ $\qquad \frac{dY}{dU} = -\frac{1}{U}\sin U - \frac{1}{U^2}\cos U$

$$\frac{dY}{dx} = -\frac{1}{x^2}\left(-x\sin\frac{1}{x} - x^2\cos\frac{1}{x}\right)$$
$$= \frac{1}{x}\sin\frac{1}{x} + \cos\frac{1}{x}$$

m) $Y = \frac{\sin 2x}{\tan x}$

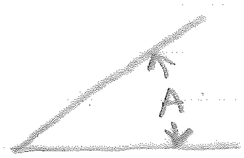
$U = \sin 2x$

$V = 2x \qquad U = \sin V$

$\frac{dU}{dx} = 2\cos 2x$

$$\frac{dY}{dx} = \frac{2\tan x\cos 2x - \sin 2x\sec^2 x}{\tan^2 x}$$
$$= \frac{2\tan x(2\cos^2 x - 1) - (2\sin x\cos x)\sec x}{\tan^2 x}$$
$$= \frac{4\sin x\cos x - 2\tan x - 2\sin x}{\tan^2 x}$$

11)



$R = \frac{V^2}{16}\cos A\sin A$

$\frac{dR}{dA} = \frac{V^2}{16}(\cos^2 A - \sin^2 A) = 0$

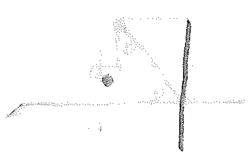
$V \neq 0$

$\cos^2 A = \sin^2 A$

$\cos A = \sin A$

$A = \frac{\pi}{4}$

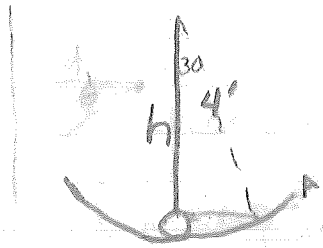
15)



$a = 32$

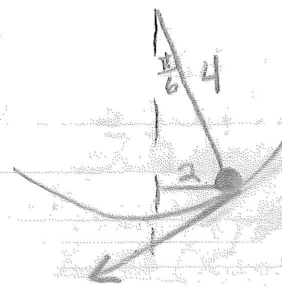
$V =$

16)

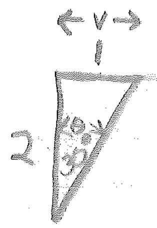


$\frac{dA}{dt} = \frac{4\pi}{3} \; \frac{1}{10} \; \frac{1}{4} = \frac{\pi}{30}$

$h =$



17)



$\frac{d\theta}{dt} = 15°/min = \frac{\pi}{12}/min$

$\frac{dv}{dt} = \frac{dv}{d\theta}\frac{d\theta}{dt}$

$v = 2\tan\theta$

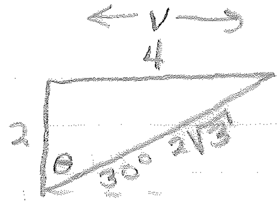
$\frac{dv}{d\theta} = 2\sec^2\theta$

$\frac{dv}{dt} = 2\sec^2\theta \, \frac{d\theta}{dt}$

$\frac{dv}{dt} = \frac{\pi}{6}\sec^2\theta$

$= \frac{\pi}{6}\left(\frac{\sqrt{3}}{2}\right)^2 = \frac{\pi}{6}\,\frac{3}{4} = \frac{3\pi}{10}$

17)



$\frac{d\theta}{dt} = \frac{\pi}{12}$

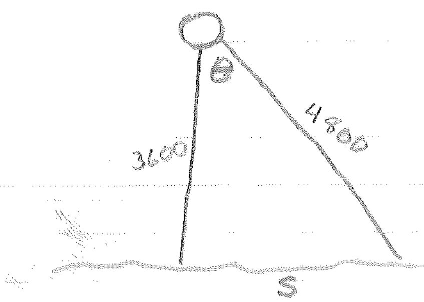
$V = 2\tan\theta$

$\frac{dV}{d\theta} = 2\sec^2\theta$

$\frac{dV}{dt} = \frac{2\pi}{12}\left(\frac{\sqrt{3}}{1}\right)^2 = \frac{6\pi}{12} = \frac{\pi}{2}$

18)

$\frac{d\theta}{dt} = 4\pi$



$s = 3600\tan\theta$

$\frac{ds}{d\theta} = 3600\sec^2\theta$

$\frac{ds}{dt} = 14{,}400\pi\sec^2\theta$

a) $\theta = 0$

$\frac{ds}{dt} = 14{,}400$

b) $\sec\theta = \frac{48}{36} = \frac{4}{3}$

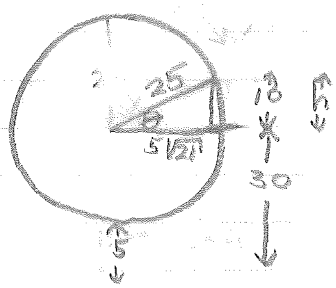
$\frac{ds}{dt} = 14{,}400\pi\left(\frac{16}{9}\right)$

$= 1600\pi(16)$

$= 25{,}600\pi$

19)

$\frac{d\theta}{dt} = \pi$



$h = 25\sin\theta$

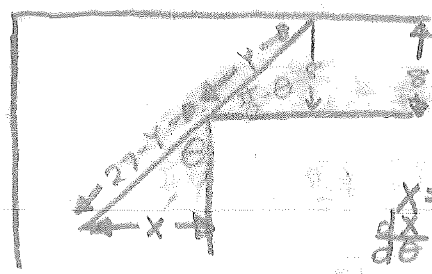
$\frac{dh}{d\theta} = 25\cos\theta$

$\frac{dh}{dt} = 25\pi\cos\theta$

$\cos\theta = \frac{\sqrt{21}}{5}$

$\frac{dh}{dt} = 5\pi\sqrt{21}$

21)



$X = (27-Y)(\sin\theta)$

$Y = 8\csc\left(\frac{\pi}{2}-\theta\right)$

$= 8\sec\theta$

$X = (27-8\sec\theta)(\sin\theta)$

$\frac{dX}{d\theta} = (27-8\sec\theta)(\cos\theta) - \sin\theta\,(8\sec\theta\tan$

$27\cos\theta - 8 - 8(\tan^2\theta) = 0$

$7\cos\theta - 1 - 8\tan^2\theta = 0$

$7\cos\theta = 1 + \tan^2\theta$

$7\cos\theta = \sec^2\theta$

$7 = \sec^3\theta$

$\sec\theta = \sqrt[3]{7}$

PROVE

if $Y = \sin X$ then $\dot{Y} = \cos X$

$Y = \sin X$

$Y + \Delta Y = \sin(X+\Delta X)$

$\Delta Y = \sin(X+\Delta X) - \sin X$

$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

$\Delta Y = 2\cos\frac{2X+\Delta X}{2}\sin\frac{\Delta X}{2}$

$\frac{\Delta Y}{\Delta X} = \cos\frac{2X+\Delta X}{2}\left[\frac{\sin\frac{\Delta X}{2}}{\Delta X/2}\right]$

$\frac{dY}{dX} = \dot{Y} = \cos X$

17)

$\frac{dx}{dt} = \frac{\pi}{12}$

$\frac{ds}{dt} = \frac{dx}{dt}\,\frac{ds}{dx}$

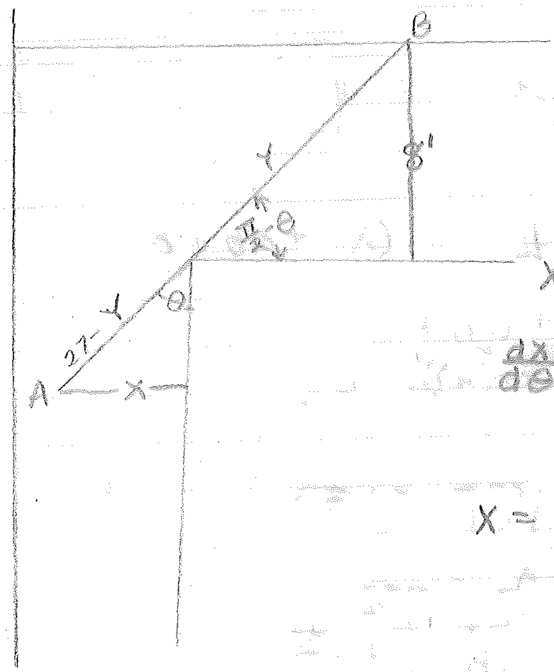
$s = 2\cot x$

$\frac{ds}{dt} = -2\csc^2 x \qquad x = \frac{\pi}{6}$

$\frac{ds}{dt} = -2\csc x\,\frac{dx}{dt}$

$= -2(4)\frac{\pi}{12} = \frac{8\pi}{12} = \frac{2\pi}{3}$

21)



$X = (27-Y)\sin\theta$

$Y = \sec\left(\frac{\pi}{2}-\theta\right)8$

$= 8\csc\theta$

$X = (27-8\csc\theta)\sin\theta$

$= 27\sin\theta - 8$

$\frac{dx}{d\theta} = 27\cos\theta = 0$

$\cos\theta = 0$

$\theta = \pi/2$

$X = (27-Y)\sin\theta$

$\sin\theta = 1 \Rightarrow X = 27-Y$

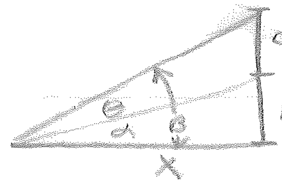
$Y = \sec\left(\frac{\pi}{2}-\theta\right)8$

$= 8\csc\theta$

$\csc\theta = 1$

$Y = 8$

$X = 27-8 = 19$



$$x = b\cot\alpha \qquad x = (a+b)\cot\beta$$

$$\frac{1}{x} = \frac{1}{b}\tan\alpha = \frac{1}{a+b}\tan\beta$$

$$\tan\beta - \tan\alpha = \tan\theta = \frac{\tan\beta - \tan\alpha}{1+\tan\alpha\tan\beta}$$

$$\tan\alpha = \frac{b}{x} \qquad \tan\beta = \frac{a+b}{x}$$

$$\tan\theta = \frac{\frac{a+b}{x} - \frac{b}{x}}{1 + \frac{b(a+b)}{x^2}} = \frac{\frac{a}{x}}{\frac{x^2+ab+b^2}{x^2}} = \frac{ax}{x^2+ab+b^2}$$

$$\frac{d(\tan\theta)}{dx} = \frac{d(\tan\theta)}{d\theta}\frac{d\theta}{dx}$$

$$= \sec^2\theta\left[\frac{(x^2+ab+b^2)a - ax(2x)}{(x^2+ab+b^2)^2}\right]$$

$$0 = \sec^2\theta\left[\frac{ax^2+a^2b+ab^2-2ax^2}{(x^2+ab+b^2)^2}\right]$$

$$x^2+ab+b^2-2x^2 = 0$$

$$-x^2 = -ab-b^2$$

$$x^2 = ab+b^2$$

$$x = \sqrt{ab+b^2}$$

$$\tan\theta = \frac{ax}{x^2+ab+b^2}$$

$$= \frac{a(ab+b^2)^{\frac{1}{2}}}{ab+b^2+ab+b^2}$$

$$= \frac{a(ab+b^2)^{\frac{1}{2}}}{2(ab+b^2)} = \frac{a}{2(ab+b^2)^{\frac{1}{2}}}$$

WORKSHEET

1) $Y = \sin^3 X$

$\dot{Y} = \sin X \cos^2 X + \cos^2 X \cos X$

$U = \cos^2 X$

$\dot{U} = {}^-2 \sin X \cos X = {}^-\sin^2$

$\sin 2X = 2 \sin X \cos X$

$\dot{Y} = \cos^3 X - \sin^3 X$

2) $Y = \sin(3X+2)$

$U = 3X+2 \quad Y = \sin U$

$\dot{Y} = 3\cos(3X+2)$

3) $Y = \sin^4 2X$

$U = 2X \qquad Y = \sin^4 U$

$\frac{dY}{dU} = 2\sin^3 U \, \dot{\sin^2 U}$

$V = \sin^2 U$

$\frac{dV}{dU} = 2\cos U \sin U$

$\frac{dY}{dU} = 4 \sin^3 U \cos U$

$\frac{dY}{dX} = 8 \sin^3 2X \cos 2X$

5) $Y = X \sin X$

$\frac{dY}{dX} = X \cos X + \sin X$

6) $Y = X^2 \sin X$

$\frac{dY}{dX} = X^2 \cos X + 2X \sin X$

7) $Y = (1 - \sin X)^{\frac{1}{2}}$

$U = 1 - \sin X \qquad Y = U^{\frac{1}{2}}$

$\frac{dY}{dX} = {}^-\cos X \left[\frac{1}{2}(1-\sin X)\right]^{-\frac{1}{2}}$

$= \frac{{}^-\cos X}{2\sqrt{1-\sin X}}$

8) $Y = (\sin X)^{\frac{1}{2}}$

$U = \sin X \qquad Y = U^{\frac{1}{2}}$

$\frac{dY}{dX} = \cos X \left[\frac{1}{2}(\sin X)^{-\frac{1}{2}}\right]$

9) $s = a(t - \sin t) = at - a\sin t$

$\frac{ds}{dt} = a - a\cos t$

10) $Y = \frac{x^2}{\sin X}$

$\frac{dY}{dX} = \frac{2X \sin X + X^2 \cos X}{\sin^2 X}$

11) $U = \frac{z}{\sin 4z} = z \csc 4z$

$\frac{du}{dz} = -4z \csc 4z \cot 4z + \csc 4z$

13) $Y = \cos^2 3X$

$U = 3X \qquad Y = \cos^2 U$

$\frac{du}{dx} = 3 \qquad \frac{dY}{dU} = -2 \sin U \cos U$

$\frac{dY}{dX} = -6 \sin 3X \cos 3X$

12) $Y = \cos(2X-3)$

$U = 2X-3 \qquad Y = \cos U$

$\frac{du}{dx} = 2 \qquad \frac{dY}{dU} = -\sin U$

$\frac{dY}{dX} = -2 \sin(2X-3)$

14) $Y = 2\cos^2 \frac{1}{2}X$

$U = \frac{1}{2}X \qquad Y = 2\cos^2 U$

$\frac{du}{dx} = \frac{1}{2} \qquad \frac{dY}{dU} = -4 \cos U \sin U$

$\frac{dY}{dX} = -2 \cos \frac{1}{2}X \sin \frac{1}{2}X$

15) $U = 2 \sin X + 3 \cos X$

$\frac{du}{dx} = 2 \cos X - 3 \sin X$

16) $w = \sin 2t \cos 3z$

$\frac{dw}{dt} = -\sin 2t(3 \sin 3z) + \cos 3z(2\cos 2z)$

$= 2\cos 3z \cos 2z - 3 \sin 2z \sin 3z$

17) $s = t \sin t + \cos 2t$

$\frac{ds}{dt} = t\cos t + \sin t - 2 \sin 2t$

18) $s = t^2 \cos 2t$

$\frac{ds}{dt} = -2t^2 \sin 2t + 2t \cos 2t$

19) $Y = \tan 2X$

$\frac{dY}{dX} = 2 \sec^2 2X$

20) $Y = 3 \cot(X-1)$

$\frac{dY}{dX} = -3 \csc^2(X-1)$

21) $Y = 2 \tan \frac{1}{2}X$

$\frac{dY}{dX} = \sec^2 \frac{1}{2}X$

22) $y = \sec(x+2)$
$\frac{dy}{dx} = \sec x + 2 \tan x + 2$

23) $f(x) = \frac{1}{2} \sec 2x$
$f'(x) = \sec 2x \tan 2x$

24) $w = \cot(1-2z)$
$\frac{dw}{dz} = 2 \csc^2 (1-2z)$

25) $y = \csc(4x-3)$
$\frac{dy}{dx} = {}^-4 \csc(4x-3) \cot(4x-3)$

26) $y = \frac{1}{2} \csc(2x-3)$
$\frac{dy}{dx} = {}^- \csc(2x-3) \cot(2x-3)$

27) $y = \frac{1}{4} \tan t^2$
$\frac{dy}{dt} = \frac{1}{2} t \sec^2 t^2$

28) $s = 2 \sec t^3$
$\frac{ds}{dt} = 6t^2 \sec t^3 \tan t^3$

29) $y = \sec z^{\frac{1}{2}}$
$\frac{dy}{dz} = \frac{1}{2} z^{-\frac{1}{2}} \sec z^{\frac{1}{2}} \tan z^{\frac{1}{2}}$

30) $f(x) = \tan (x+1)^{\frac{1}{2}}$
$f'(x) = \frac{1}{2\sqrt{x+1}} \sec^2 (x+1)^{\frac{1}{2}}$

31) $y = x \tan x$
$\frac{dy}{dx} = x \sec^2 x + \tan x$

32) $y = (\tan x) - x$
$\frac{dy}{dx} = (\sec^2 x) - 1$

33) $y = \sec^2 x$
$\frac{dy}{dx} = 2 \sec x \sec x \tan x$
$= 2 \sec^2 x \tan x$

34) $y = x \tan^2 x$
$\frac{dy}{dx} = 2x \sec^2 x \tan x + \tan^2 x$

35) $f(t) = \sec 2t - \tan 2t$
$f'(t) = 2 \sec 2t \tan 2t - 2 \sec^2 2t$

36) $y = \sin x \tan x$
$\frac{dy}{dx} = \sin x \sec^2 x + \cos x \tan x$
$= \sin x \sec^2 x + \sin x$

37) $V = \tan^3 2t$

$U = 2t \qquad V = \tan^3 U$

$\frac{dV}{dU} = 3\tan^2 U \sec^2 U$

$\frac{dV}{dt} = 6\tan^2 2t \sec^2 2t$

38) $s = \tan\left(\frac{\pi}{4} - \frac{t}{2}\right)$

$\frac{ds}{dt} = -\frac{1}{2}\sec^2\left(\frac{\pi}{4} - \frac{t}{2}\right)$

39) $w = z + \cot z$

$\frac{dw}{dz} = 1 - \csc^2 z$

40) $w = \frac{z^2}{\cos z} = z^2 \sec z$

$\frac{dw}{dz} = z^2 \sec z \tan z + 2z \sec z$

41) $Y = \frac{\tan x}{x^2} = (\tan x)\, x^{-2}$

$\frac{dY}{dx} = \tan x\,(-2x^{-3}) + x^{-2}\sec^2 x$

$= -\frac{2}{x^3}\tan x + \frac{\sec^2 x}{x^2}$

42) $Y = \sec 2x \tan 2x$

$\frac{dy}{dx} = 2\sec 2x \sec^2 2x + 2\tan 2x \sec 2x \tan 2x$

$= 2\sec^3 2x + 2\tan^2 2x \sec 2x$

Pg 247

1) a) $\dot{Y} = \cos 3x$

$U = 3x \qquad \dot{Y} = \cos U$

$\frac{dU}{dx} = 3 \qquad Y = \sin U$

$Y = \frac{\sin 3x}{3} + C$

c) $\dot{Y} = \sin^2 x$

$U = 2x \qquad \frac{dU}{dx} = 2$

$\dot{Y} = \sin^2 \frac{U}{2} = \frac{1}{2} - \frac{\cos U}{2}$

$= \frac{1}{2} - \frac{1}{2}\cos U$

$Y = \left(\frac{x}{2} - \frac{1}{2}\sin U\right)\frac{du}{dx}$

$= \frac{x}{4} - \frac{1}{4}$

c) $\mathring{Y} = \sin^2 X$

$U = 2X$

$\mathring{Y} = \sin^2 \frac{U}{2} = \frac{1}{2} - \frac{1}{2}\cos U$

$Y = \left(\frac{U}{2} - \frac{1}{2}\sin U\right)\frac{1}{2} + C$

$= \left(\frac{2X}{2} - \frac{1}{2}\sin 2X\right)\frac{1}{2} + C$

$= \frac{X}{2} - \frac{1}{4}\sin 2X + C$

$= \frac{X}{2} - \frac{1}{4}(2\sin X\cos X) + C$

$= \frac{X}{2} - \frac{1}{2}\sin X\cos X$

~~e)~~ $\mathring{Y} = \sin^2 X \cos X$

$= (1-\cos^2 X)\cos X$

$= \cos X - \cos^3 X$

$\frac{dY}{dX} = \sin X - \cos^3 X$

$U = {}^-\cos^3 X$

$= {}^-(1-\sin^2 X)\cos X$

$= {}^-\cos X - \sin^2 X\cos X$

$\frac{dU}{dX} = {}^-\sin X + \sin^3 X\cos X$

$\frac{dY}{dX} = \sin X - \sin X - \sin^2 X\cos X$

Ex) $\mathring{Y} = \cos 2X$

$U = 2X \quad \frac{dU}{dX} = 2$

$\mathring{Y} = \frac{1}{2}\sin U = \frac{1}{2}\sin 2X + C$

e) $\mathring{Y} = \sin^2 X\cos X$

$U = \sin X$

$\frac{dU}{dX} = 2\sin X\cos X$

$\mathring{Y} = U^2\cos X$

$\mathring{Y} = -U^2(-\cos X) = -U^2\frac{dU}{dX}$

$= -\frac{U^3}{3} + C$

$= -\frac{\sin^3 X}{3} + C$

~~g)~~ $\sec^3 x \tan x = \dot{y}$

$U = \sec^2 x$

$\frac{du}{dx} = \tan x$

$\dot{y} = U^{\frac{3}{2}} \tan x = U^{\frac{3}{2}} \frac{du}{dx}$

$= U^{\frac{5}{2}}$

$Y = \frac{2}{7} \sec^2 x^{\frac{7}{2}}$

$= \frac{2}{7} \sec^7 x + C$

~~g)~~ $\dot{y} = \sec^3 x \tan x$

$U = \sec^2 x$

$\frac{du}{dx} = \tan x$

$\dot{y} = U^{\frac{3}{2}} \tan x = U^{\frac{3}{2}} \frac{du}{dx}$

$= U^{\frac{5}{2}}$

$Y = \frac{2}{3} \sec^2 x^{\frac{3}{2}}$

g) $\dot{y} = \sec^3 x \tan x$

$U = \sec x$

$\frac{du}{dx} = \sec x \tan x$

$\dot{y} = U^3 \tan x$

$= U^3 \frac{du}{dx} \frac{1}{\sec x}$

$= U^3 \frac{du}{dx} U^{-1}$

$= U^2 \frac{du}{dx}$

$\frac{dY}{dx} = \frac{1}{3} U^3 + C$

$= \frac{\sec^3 x}{3} + C$

1) $y' = \tan^3 2x \sec^2 2x$

$U = \tan 2x$

$V = 2x \quad \frac{dV}{dx} = 2$

$\frac{du}{dx} = 2\sec^2 2x$

$y' = U^3 \sec^2 2x = \frac{1}{2} U^3 \frac{du}{dx}$

$y = \frac{U^4}{8} + C = \frac{\tan^4 2x}{8} + C$

k) $y' = \cot^{\frac{3}{2}} x \csc^2 x$

$U = \cot x$

$\frac{du}{dx} = -\cot x \csc x$

$y' = U^{\frac{3}{2}} \csc^2 x = U^{\frac{3}{2}} \frac{du}{dx} \left(\frac{1}{\cot^2 x}\right)$

$= -U^{\frac{3}{2}} \cot x \csc x \frac{1}{\cot^2} \frac{du}{dx}$

$= -U^{\frac{3}{2}} \frac{\csc x}{\cos x} \frac{du}{dx} = -U^{\frac{3}{2}} \frac{\tan x}{\sin x} \frac{du}{dx}$

$= -U^{\frac{3}{2}} \sec x \frac{du}{dx}$

~~k)~~ $y' = \cot^{\frac{3}{2}} x \csc^2 x$

$U = \csc x$

$\frac{du}{dx} = -\csc x \cot x$

$y' = \cot^{\frac{3}{2}} x \, U^2 = -U^2 \frac{du}{dx} \cot^{\frac{1}{2}} x \sin x$

$= -U^2 \frac{du}{dx} \cos^{\frac{1}{2}} x \sin^{\frac{1}{2}} x$

~~k)~~ $y' = \cot^{\frac{3}{2}} x \csc^2 x$

$U = \cot^3 x$

$\frac{du}{dx} = -3\cot^2 x \csc x$

$y' = U^{\frac{1}{2}} \csc^2 x = -\frac{1}{3} U^{\frac{1}{2}} \frac{du}{dx} \frac{\csc x}{\cot^2 x}$

$= -\frac{1}{3} U^{\frac{1}{2}} U^{-\frac{2}{3}} \csc x$

$= -\frac{1}{3} U^{-\frac{1}{3}} \csc x$

m) $\dot{y} = \sec^5 x \tan x$

$u = \sec x$

$\frac{du}{dx} = \tan x \sec x$

$\dot{y} = u^4 \cancel{\sec x \tan x}$

$= u^4 \frac{du}{dx}$

$y = \frac{\sec^5 x}{5} + C$

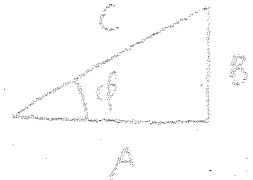
k) $\dot{y} = \cot^{\frac{3}{2}} x \csc^2 x$

$u = \cot x$

$\frac{du}{dx} = -\csc^2 x$

$\dot{y} = -u^{\frac{3}{2}} \frac{du}{dx}$

$y = -\frac{2}{5} u^{\frac{5}{2}} + C = -\frac{2}{5} \cot^{\frac{5}{2}} x + C$



If $C = \sqrt{A^2+B^2}$, then C can never be negative.

$-\sin\theta = +\sin(\theta+\pi) = +\sin(\theta-\pi)$

(SEE PAGE 28)

A	B	y (52)	C	ϕ	y (55)
0	+	$B\cos\sqrt{\frac{k}{m}}\,x$	B	$\frac{\pi}{2}$	$B\sin\left(\sqrt{\frac{k}{m}}\,x+\frac{\pi}{2}\right) = -B\sin\left(\sqrt{\frac{k}{m}}\,x+\frac{3\pi}{2}\right)$
0	−	$B\cos\sqrt{\frac{k}{m}}\,x$	$-B$	$-\frac{\pi}{2}$	$-B\sin\left(\sqrt{\frac{k}{m}}\,x-\frac{\pi}{2}\right) = B\sin\left(\sqrt{\frac{k}{m}}\,x+\frac{\pi}{2}\right)$
+	0	$A\sin\sqrt{\frac{k}{m}}\,x$	A	0	$A\sin\left(\sqrt{\frac{k}{m}}\,x+0\right) = -A\sin\left(\sqrt{\frac{k}{m}}\,x+\pi\right)$
−	0	$A\sin\sqrt{\frac{k}{m}}\,x$	$-A$	π	$-A\sin\left(\sqrt{\frac{k}{m}}\,x+\pi\right) = A\sin\left(\sqrt{\frac{k}{m}}\,x+2\pi\right)$

$\phi = \sin^{-1}\frac{B}{C}$

$\phi = \cos^{-1}\frac{A}{C}$

$\phi = \tan^{-1}\frac{B}{A}$

quadrant

	I	II	III	IV
sin	+	+	−	−
cos	+	−	−	+
tan	+	−	+	−

In general,

$y = A\sin\sqrt{\frac{k}{m}}\,x + B\cos\sqrt{\frac{k}{m}}\,x$

A = initial velocity $\times\sqrt{\frac{m}{k}}$

B = initial displacement from equilibrium position

$y = C\sin\left(\sqrt{\frac{k}{m}}\,x+\phi\right)$

C = amplitude

ϕ = phase angle

DRILL PROBLEMS INVOLVING DIFFERENTIATION OF INVERSE TRIG FUNCTIONS

INSTRUCTIONS: DIFFERENTIATE EACH OF THE FOLLOWING

1. $y = \sin^{-1} 2x$
2. $y = \sin^{-1}(1-x)$
3. $y = \tan^{-1}(x-1)$
4. $y = \tan^{-1}\left(\frac{1}{x}\right)$
5. $w = \tan^{-1} z$
6. $f(x) = x^2 \tan^{-1}(x^2)$
7. $u = \sin^{-1}\sqrt{x}$
8. $v = \sin^{-1} \frac{z+1}{\sqrt{2}}$
9. $y = \sec^{-1}(2-x)$
10. $y = \sec^{-1}(x^2)$
11. $F(x) = x \cot^{-1} \frac{x}{2}$
12. $y = x^2 \cos^{-1} x$
13. $w = \frac{1}{z} \tan^{-1} z$
14. $f(x) = \frac{1}{x} \cot^{-1} 2x$
15. $s = t^2 \csc^{-1}\sqrt{t}$
16. $u = \csc^{-1}(z^2)$
17. $y = a^2 \sin^{-1} \frac{x}{a} - x\sqrt{a^2 - x^2}$
18. $y = x \sin^{-1} x + \sqrt{1-x^2}$
19. $y = (x^2+1)\tan^{-1} x - x$
20. $y = \frac{x}{\sqrt{a^2-x^2}} - \sin^{-1}\frac{x}{a}$

FIND THE SECOND DERIVATIVE OF

21. $y = \sin^{-1} \frac{x}{2}$
22. $y = x \tan^{-1} x$

[illegible] DIFFERENTIATION

[illegible] INVERSE TRIG FUNCTIONS

1. $\dfrac{2}{\sqrt{1-4x^2}}$

3. $\dfrac{1}{x^2-2x+2}$

5. $\dfrac{z}{1+z^2}+\tan^{-1} z$

7. $\dfrac{1}{2\sqrt{x(1-x)}}$

9. $\dfrac{-1}{\sqrt{(2-x)^2(3-4x+x^2)}}$

11. $\dfrac{-2x}{4+x^2}+\cot^{-1}\dfrac{x}{2}$

13. $\dfrac{1}{z(1+z^2)}-\dfrac{1}{z^2}\tan^{-1} z$

15. $\dfrac{-x^2}{2\sqrt{x^2(x-1)}}+2x\cos^{-1}\sqrt{x}$

17. $\dfrac{2x^2}{[illegible]}$

19. $2x\tan^{-1} x$

21. $\dfrac{x}{(4-x^2)^{3/2}}$

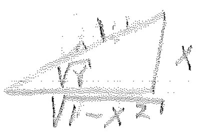
Pg 280

e) $Y = (\sin^{-1} x)^2$

$Y = (\sin^{-1})^2 x^2$

$x^2 = \sin^2 Y$

$x = \sin \sqrt{Y}$



$\frac{dx}{dY} = \frac{1}{2\sqrt{Y}} \cos \sqrt{Y}$

$\frac{dY}{dx} = 2\sqrt{Y} \sec \sqrt{Y}$

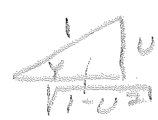
$= 2 \sin^{-1}\sqrt{x} \Big/ \sqrt{1-x^2}$

g) $Y = \sin^{-1}(\cos x)$

$U = \cos x \quad \frac{du}{dx} = -\sin x$

$Y = \sin^{-1} U$

$\sin Y = U$



$\frac{du}{dx} = \cos Y$

$\frac{dY}{du} = \sec Y = \frac{1}{\sqrt{1-U^2}}$

$\frac{dY}{dx} = \frac{1}{1-U^2} \cdot \frac{1-U^2}{1} = 1$

WORKSHEET

1) $Y = \sin^{-1} 2x$

$\sin Y = 2x$

$x = \frac{1}{2} \sin Y$

$\frac{dx}{dY} = \frac{1}{2} \cos Y = \frac{\sqrt{1-4x^2}}{2}$

$\frac{dY}{dx} = \frac{2}{\sqrt{1-4x^2}}$

2) $Y = \tan^{-1}(x-1)$

$\tan Y = x - 1$

$x = \tan Y + 1$

$\frac{dx}{dY} = \sec^2 Y$

$\frac{dY}{dx} = \cos^2 Y = \quad 2x - x^2$

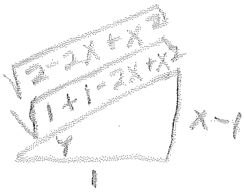
3) $Y = \tan^{-1}(X-1)$

$\tan Y = X - 1$

$X = \tan Y + 1$

$\frac{dX}{dY} = \sec^2 Y$

$\frac{dY}{dX} = \cos^2 Y = \frac{1}{2 - 2X + X^2}$

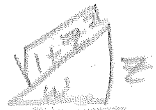


5) $W = \tan^{-1} Z$

$\tan W = Z$

$\frac{dZ}{dW} = \sec^2 W$

$\frac{dW}{dZ} = \cos^2 W = \frac{1}{1+Z^2}$



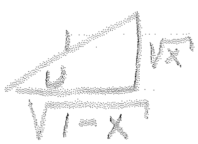
7) $U = \sin^{-1}\sqrt{X}$

$\sqrt{X} = \sin U$

$X = \sin^2 U$

$\frac{dX}{dU} = 2\sin U \cos U$

$\frac{dU}{dX} = \frac{1}{2}\sec U \csc U = \frac{1}{2\sqrt{X}\sqrt{1-X}}$



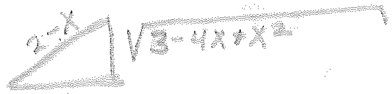
9) $Y = \sec^{-1}(2-X)$

$2 - X = \sec Y$

$X = 2 - \sec Y$

$\frac{dX}{dY} = -\sec Y \tan Y$

$\frac{dY}{dX} = -\cos Y \cot Y = \frac{1}{(2-X)\sqrt{3-4X+X^2}}$



11) $F(X) = X\cot^{-1}\frac{X}{2}$

Let $Y = F(X)$

$Y = X\cot^{-1}\frac{X}{2}$

$U = \cot^{-1}\frac{X}{2}$

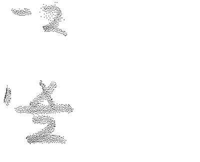
$\cot U = \frac{X}{2}$

$X = 2\cot U$

$\frac{dX}{dU} = -2\csc^2 U = \frac{X^2+4}{-2}$

$\frac{dU}{dX} = \frac{-2}{X^2+4}$

$\frac{dY}{dX} = \frac{-2X}{X^2+4} + \cot^{-1}\frac{X}{2}$



13) $w = \frac{1}{z} \tan^{-1} z$

$U = \tan^{-1} z$

$z = \tan U$

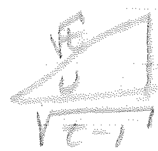
$\frac{dz}{dU} = \sec^2 U$

$\frac{dU}{dz} = \cos^2 U = \frac{1}{z^2+1}$

$\frac{dw}{dz} = \frac{1}{z^3+z} - \frac{1}{z^2} \tan^{-1} z$

15) $s = t^2 \csc^{-1} \sqrt{t}$

$U = \csc^{-1} \sqrt{t}$



$\sqrt{t} = \csc U$

$t = \csc^2 U$

$\frac{dt}{dU} = -2 \csc U \cot U$

$\frac{dU}{dt} = -\frac{1}{2} \sin U \tan U = -\frac{1}{2\sqrt{t}\sqrt{t-1}}$

$\frac{ds}{dt} = -\frac{t^2}{2\sqrt{t^2-t}} + 2t \csc^{-1} \sqrt{t}$

17) $Y = a^2 \sin^{-1} \frac{x}{a} - x\sqrt{a^2 - x^2}$

$U = \sin^{-1} \frac{x}{a}$

$\sin U = \frac{x}{a}$

$x = a \sin U$

$\frac{dx}{dU} = a \cos U = \sqrt{a^2 - x^2}$

$\frac{dU}{dx} = \frac{1}{\sqrt{a^2-x^2}}$

$V = \sqrt{a^2 - x^2}$

$d = a^2 - x^2$

$V = d^{\frac{1}{2}}$

$\frac{dV}{dx} = -\frac{x}{\sqrt{a^2-x^2}}$

$\frac{dY}{dx} = \frac{a^2}{\sqrt{a^2-x^2}} - \sqrt{a^2-x^2} - \frac{x^2}{\sqrt{a^2-x^2}}$

$= \frac{a^2-x^2}{\sqrt{a^2-x^2}} - \sqrt{a^2-x^2} = 0$

17) $Y = a^2 \sin^{-1}\frac{x}{a} - x\sqrt{a^2 - x^2}$

$U = a^2 \sin^{-1}\frac{x}{a}$

$\frac{U}{a^2} = \sin^{-1}\frac{x}{a}$

$\sin \frac{U}{a^2} = \frac{x}{a}$

$\sin U \cos a^2 + \sin \frac{1}{a^2} \cos U = \frac{x}{a}$

21) $Y = \sin^{-1}\frac{x}{2}$

$\sin Y = \frac{x}{2}$

$X = 2 \sin Y$

$\frac{dx}{dY} = 2\cos Y = \sqrt{4-x^2}$

$\frac{dY}{dx} = \frac{1}{\sqrt{4-x^2}}$

$U = 4 - x^2 \quad \frac{dY}{dx} = U^{-\frac{1}{2}}$

$\frac{d^2Y}{dx^2} = -2x\left[-\frac{1}{2}(4-x^2)^{-\frac{3}{2}}\right]$

$= x(4-x^2)^{-\frac{3}{2}}$

Pg 381

1) a) $\lim\limits_{x \to 2} \frac{x^2-4}{x^2-5x+6} = \lim\limits_{x \to 2} \frac{2x}{2x-5} = \frac{4}{-1} = -4$

b) $\lim\limits_{x \to a} \frac{x-a}{x^3-a^3} = \lim\limits_{x \to a} \frac{1}{3x^2} = \frac{1}{3a^2}$

c) $\lim\limits_{x \to n} \frac{\log \frac{x}{n}}{n-x} = \lim\limits_{x \to n} \frac{\log x - \log n}{n-x} = \frac{\frac{1}{x}}{-1} = \frac{-1}{n}$

e) $\lim\limits_{x \to 0} \frac{xe^x - x}{1-\cos 2x} = \lim\limits_{x \to 0} \frac{xe^x + e^x - 1}{+2\sin 2x} = \lim \frac{xe^x + 2e^x}{4\cos 2x} = \frac{1}{2}$

g) $\lim\limits_{x \to 0} \frac{\tan x - x}{x - \sin x} = \lim\limits_{x \to 0} \frac{\sec^2 x - 1}{1-\cos x} = \lim\limits_{x \to 0} \frac{\tan^2 x}{1-\cos x} =$

$\lim\limits_{x \to 0} \frac{2\tan x \sec^2 x}{1+\sin x}$

g) $\lim\limits_{x \to 0} \frac{\tan x - x}{x - \sin x} = \lim\limits_{x \to 0} \frac{\sec^2 x - 1}{1-\cos x} = \lim\limits_{x \to 0} \frac{\tan^2 x}{1-\cos x} =$

$\lim\limits_{x \to 0} \frac{\tan x \sec^2 x}{\sin x} = \lim\limits_{x \to 0} \sec^3 x = 1$

i) $\lim\limits_{x \to 5} \frac{2-(x-1)^{\frac{1}{2}}}{x^2-25} = \lim\limits_{x \to 5} \frac{-1}{4x\sqrt{x-1}} = \frac{-1}{40}$

j) $\lim\limits_{x \to 0} \frac{\tan x}{\tan 4x} = \lim\limits_{x \to 0} \frac{\sec^2 x}{4\sec^2 4x} = \frac{1}{4}$

4) $\lim_{K \to 0} \frac{32 - 32e^{-KT}}{K} = \lim \frac{32te^{-KT}}{1} = 32t$

1) $Y = \int x e^{-x} dx$

$dv = e^{-x} \quad u = x$

$v = -e^{-x} \quad du = dx$

$Y = -x e^{-x} + \int e^{-x} dx$

$= -x e^{-x} - e^{-x} + C$

3) $Y = \int x^2 e^{-x}$

$dv = e^{-x} \quad u = x^2$

$v = -e^{-x} \quad du = 2x\,dx$

$Y = -x^2 e^{-x} + \int 2x e^{-x} dx$

$dv = e^{-x} \quad u = 2x$

$v = -e^{-x} \quad du = 2dx$

$Y = -x^2 e^{-x} - 2x e^{-x} + \int 2 e^{-x} dx$

$= -x^2 e^{-x} - 2x e^{x} - 2e^{-x} + C$

5) $Y = \int x \sin nx\, dx$

$u = x \quad dv = \sin nx$

$du = dx \quad v = -\frac{1}{n} \cos nx$

$Y = \frac{-x}{n} \cos nx + \int \frac{1}{n} \cos nx\, dx$

$Y = \frac{-x}{n} \cos nx + \frac{1}{n^2} \sin nx + C$

6) $Y = \int t \log t$

$U = \log t \qquad dV = t$

$du = \frac{1}{t}\,dt \qquad V = \frac{t^2}{2}$

$Y = \frac{t^2}{2}\log t - \int \frac{t}{2}\,dt$

$= \frac{t^2}{2}\log t - \frac{t^2}{4} + C$

7) $Y = \int z^2 \sin z \, dz$

$dV = z^2 \qquad U = \sin z$

$V = \frac{z^3}{3} \qquad dU = \cos z \, dz$

$Y = \frac{z^3}{3}\sin z - \int \frac{z^3}{3}\cos z \, dz$

$U = z^2 \qquad dV = \sin z$

$dU = 2z\,dz \qquad V = -\cos z$

$Y = -z^2 \cos z + \int 2z \cos z$

$U = 2z \qquad dV = \cos z$

$dU = 2\,dz \qquad V = \sin z$

$Y = -z^2 \cos z + 2z \sin z - \int 2 \sin z$

$= -z^2 \cos z + 2z \sin z + 2 \cos z$

8) $Y = \int x^2 e^{-3x}\, dx$

$dV = x^2 \quad U = e^{-3x}$

$V = \frac{x^3}{3} \quad dV = -3e^{-3x}dx$

$Y = \quad dv = e^{-3x} \quad U = x^2$

$V = \frac{-1}{3}e^{-3x} \quad dU = 2x\, dx$

$Y = \frac{-x^2}{3}e^{-3x} + \int \frac{2x}{3}e^{-3x}dx$

$U = \frac{2x}{3} \quad dV = e^{-3x}$

$du = \frac{2}{3}dx \quad V = \frac{-1}{3}e^{-3x}$

$Y = \frac{-x^2}{3}e^{-3x} + \frac{2}{9}xe^{-3x} - \int \frac{2}{3}e^{-3x}$

$= \frac{-x^2}{3}e^{-3x} - \frac{2}{9}xe^{-3x} + \frac{2}{9}e^{-3x} + C$

9) $Y = \int x^2 \sin x\, dx$

$U = x^2 \quad dV = \sin x$

$dU = 2x dx \quad V = -\cos x$

$Y = -x^2\cos x + \int 2x \cos x\, dx$

$U = 2x \quad dV = \cos x$

$dU = 2dx \quad V = \sin x$

$Y = -x^2\cos x + 2x\sin x - \int 2\sin x$

$= -x^2\cos x + 2x\sin x + 2\cos x$

10) $Y = \int e^{2x} \cos^2 x \, dx$

$dV = e^{2x} \quad U = \cos^2 x$

$V = \frac{1}{2} e^{2x} \quad dU = -\sin 2x \, dx$

$Y = \frac{1}{2} e^{2x} \cos^2 x + \int \frac{1}{2} e^{2x} \sin 2x \, dx$

$U = \cancel{\tfrac{1}{2}}\sin 2x \quad dV = e^{2x}$

$dU = \cos 2x \quad V = \frac{1}{2} e^{2x}$

$Y = \frac{1}{2} e^{2x} \cos^2 x + \frac{1}{4} \sin 2x \, e^{2x} - \int \frac{1}{2} e^{2x} \cos 2x$

$Y = \int e^{2x} \cos^2 x \, dx$

$dV = e^{2x} \quad U = \cos^2 x$

$V = \frac{1}{2} e^{2x} \quad dU = -\sin 2x \, dx$

$Y = \frac{1}{2} e^{2x} \cos^2 x + \int \frac{1}{2} e^{2x} \sin 2x$

$U = \sin 2x \quad dV = \frac{1}{2} e^{2x}$

$dU = 2\cos 2x \quad V = \frac{1}{4} e^{2x}$

$Y = \frac{1}{2} e^{2x} \cos^2 x + \frac{1}{4} e^{2x} \sin 2x + \int \frac{-1}{2} e^{2x} \cos 2x$

$U = \cos 2x \quad dU = -\frac{1}{2} e^{2x}$

14) $Y = \int \frac{x^{-2}}{\sqrt{(a^2 - x^2)}} dx$

$dV = x^{-2} \quad U = (a^2 - x^2)^{-\frac{1}{2}}$

$V = -x^{-1} \quad dU = -2x\left(\frac{-1}{2}(a^2 - x^2)^{-\frac{3}{2}}\right) dx$

$= x(a^2 - x^2)^{-\frac{3}{2}} dx$

$Y = \frac{1}{x\sqrt{a^2 - x^2}} + \int (a^2 - x^2)^{-\frac{3}{2}} dx$

$(a^2 - x^2)^{-\frac{1}{2}}$

15) $Y = \int e^{\log \sqrt{x}}$

pg 398

1) a) $\int \frac{1}{(x-3)(x+2)} dx$

$\frac{1}{(x-3)(x+2)} \equiv \frac{A}{(x-3)} + \frac{B}{(x+2)}$

$A(x+2) + B(x-3) \equiv 1$

$A(x+2) \equiv 1 - B(x-3)$

$x = -2 \Rightarrow B = -\frac{1}{5}$

$x = 3 \Rightarrow A = \frac{1}{5}$

$\int \frac{1}{5x-15} - \frac{1}{5x+10} dx$

$\frac{1}{5}\log(5x-15) - \frac{1}{5}\log(5x+10)$

$= \frac{1}{5}\log\frac{x-3}{x+2}$

c) $\int \frac{dx}{x^2-5x+6} = \int \frac{dx}{(x-2)(x-3)}$

$$\frac{1}{(x-2)(x-3)} \equiv \frac{A}{x-2} + \frac{B}{(x-3)}$$

$A(x-3) \equiv 1 - B(x-2)$

$x = 3 \Rightarrow B = 1$

$x = {}^{-}2 \Rightarrow A = \frac{-1}{5}$

$\int \frac{1}{x-3} - \frac{1}{5x-10}$

$\log x-3 - \frac{1}{5}\log 5x-10$

d) $\int \frac{x+2}{(x-3)(x-4)}\,dx$

$$\frac{x+2}{(x-3)(x-4)} = \frac{A}{(x-3)} + \frac{B}{x-4}$$

$x+2 \equiv A(x-4) + B(x-3)$

$x = 4 \Rightarrow B = 6$

$x = 3 \Rightarrow A = {}^{-}5$

$\int \frac{6}{x-4} - \int \frac{5}{x-3}$

$6\log(x-4) - 5\log(x-3) + C$

8) $\int \frac{dx}{\cos x(1+\cos x)}$

$$\frac{1}{\cos x(1+\cos x)} = \frac{A}{\cos x} + \frac{B}{1+\cos x}$$

$A(1+\cos x) + B(\cos x) = 1$

$x = \pi \Rightarrow B = {}^{-}1$

$x = \frac{\pi}{2} \Rightarrow A = 1$

$\int \frac{1}{\cos x} - \frac{1}{1+\cos x}$

e) $\int \frac{x^2+x+1}{(x-1)(x+2)(x-3)}\,dx$

$$\frac{x^2+x+1}{(x-1)(x+2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{x+2} + \frac{C}{x-3}$$

$$x^2+x+1 = A(x+2)(x-3) + B(x-1)(x-3) + C(x-1)(x+2)$$

$x=3 \Rightarrow C = \frac{13}{10}$

$x=1 \Rightarrow A = -\frac{1}{2}$

$x=-2 \Rightarrow B = \frac{1}{5}$

$$\int \frac{1}{5(x+2)} - \frac{1}{2(x-1)} + \frac{13}{10(x-3)}$$

$$\frac{1}{5}\log(x+2) - \frac{1}{2}\log(x-1) + \frac{13}{10}\log(x-3) + C$$

WORKSHEET

19) $Y = \int \frac{dx}{x^3+1} = \int \frac{dx}{(x+1)(x^2-x+1)}$

$$\frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{(x^2-x+1)}$$

$$1 = A(x^2-x+1) + Bx + C(x+1)$$

$x=-1 \Rightarrow 1+B=A$ $\qquad 1+B=-C$

$x=0 \Rightarrow \qquad A=-C$ $\qquad B=3C+2$

$x^2-x+1=0$ $\qquad B=-3(1+B)+2$

$x = \frac{1\pm\sqrt{1-4}}{2}$ $\qquad B=-1-3B$

$= \frac{1}{2} \pm \frac{i\sqrt{3}}{2}$ $\qquad 4B=-1 \qquad B=-\frac{1}{4}$

$$1 = B\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right) + C\left(\frac{3}{2}+\frac{i\sqrt{3}}{2}\right)$$

$$1 = \frac{B}{2} + \frac{Bi\sqrt{3}}{2} + \frac{3C}{2} + \frac{Ci\sqrt{3}}{2}$$

$$2 = B + Bi\sqrt{3} + 3C + Ci\sqrt{3}$$

$$2 - B - Bi\sqrt{3} = 3C + Ci\sqrt{3} \Rightarrow B = 3C+2$$

26) $Y = \int \frac{dx}{(x+1)(x^2+1)}$

$$\frac{1}{(x+1)(x^2+1)} = \frac{A}{(x+1)} + \frac{Bx+C}{x^2+1}$$

$$1 = A(x^2+1) + (Bx+C)(x+1)$$

$$x = -1 \Rightarrow A = \frac{1}{2}$$

$$x = i$$

$$1 = Bi(1+i) + C(1+i)$$

$$1 = Bi - B + C + Ci$$

$$B - Bi = C + Ci$$

$$-Bi = Ci \qquad B = C$$

$$-B = C \qquad B = C = -B$$

$$2B = 0 \Rightarrow B = 0 \Rightarrow C = 0$$

$$Y = \int \frac{-1}{x+1}$$

$$= -\log(x+1) + C$$

LAST YEAR'S FINAL

1) a) $\int \frac{x}{\sqrt{1+x^2}}\,dx$

$\sqrt{1+x^2} + C$

b) $\int \frac{dx}{\sqrt{9-x^2}}$

$x = 3\sin\theta \quad dx = 3\cos\theta\, d\theta$

$\int \frac{3\cos\theta}{3\cos\theta}\, d\theta \qquad \theta = \sin^{-1}\frac{x}{3}$

$\theta + C$

$\sin^{-1}\frac{x}{3} + C$

c) $\int \cos^2 x\, dx$

$\int \frac{1+\cos 2x}{2}\, dx$

$\int \frac{1}{2}dx + \int \frac{\cos 2x}{2}\, dx$

$\frac{1}{2}x + \frac{1}{4}\sin 2x + C$

d) $\int x e^x\, dx$

$dv = e^x \quad u = x$

$v = e^x \quad du = dx$

$xe^x - \int e^x dx$

$xe^x - e^x + C$

2) $\int \frac{dx}{x^2-1}$

$\frac{1}{x^2-1} = \frac{A}{(x+1)} + \frac{B}{(x-1)}$

$1 \equiv A(x-1) + B(x+1)$

$x = 1 \Rightarrow B = \frac{1}{2}$

$x = -1 \Rightarrow A = -\frac{1}{2}$

$\int \frac{dx}{x^2-1} = \frac{1}{2}\ln(x-1) - \frac{1}{2}\ln(x+1)$

$= \frac{1}{2}\ln\left(\frac{x-1}{x+1}\right) + C$

b) $\int x \sin x^2 \, dx$

$-\frac{1}{2}\cos x^2 + C$

c) $\int \sin^3 2x \, dx$

$u = \sin^2 2x \quad dv = \sin 2x$

$du = 4\cos 2x \sin 2x \, dx \quad v = -\frac{1}{2}\cos 2x$

$-\frac{1}{2}\cos 2x \sin^2 2x + 2\int \cos^2 2x \sin 2x \,) \, dx$

$+ 2\int (1 - \sin^2 2x) \sin 2x \, dx$

$+ 2\int (\sin 2x - \sin^3 2x) \, dx$

$+ 2\int \sin 2x \, dx - 2\int \sin^3 2x \, dx$

$3\int \sin^3 x = -\frac{1}{2}\cos 2x \sin^2 2x - \cos 2x$

$\int \sin^3 x = -\frac{1}{6}(\cos 2x \sin^2 2x - \frac{1}{3}\cos 2x$

~~d)~~ $\int \frac{dx}{(1+x^2)^{3/2}}$

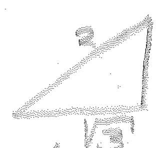
$dv = \frac{1}{1+x^2} \quad u = \cancel{(1+x^2)}^{-\frac{1}{2}}$

$v = \tan^{-1} x \quad du = x(1+x^2)^{-\frac{3}{2}}$

$\int \frac{dx}{(1+x^2)^{3/2}}$

3) a) $\int_{\pi/6}^{\pi/3} \tan^2 x \, dx$

$\int (\sec^2 x - 1) \, dx$



$\tan x - x$

$\int_{\pi/6}^{\pi/3} \tan^2 x = \tan\frac{\pi}{3} - \frac{\pi}{3} - \tan\frac{\pi}{6} + \frac{\pi}{6}$

$= \sqrt{3} - \frac{\pi}{3} - \frac{1}{\sqrt{3}} + \frac{\pi}{6}$

$= \frac{2}{\sqrt{3}} - \frac{\pi}{6}$

$= \frac{12 - \sqrt{3}\pi}{6\sqrt{3}}$

~~b)~~ $\int x\sqrt{1-x^2}\,dx$

$dv = x \qquad u = \sqrt{1-x^2}$

$v = \frac{1}{2}x^2 \quad du = \frac{-x}{\sqrt{1-x^2}}\,dx$

$\frac{1}{2}x^2\sqrt{1-x^2} + \frac{1}{2}\int \frac{x^3}{\sqrt{1-x^2}}$

c) $\int \frac{e^x}{1+e^{2x}}\,dx$

$dv = e^x \qquad u = \frac{1}{1+e^{2x}}$

$du = 2e^{2x}/(1+e^{2x})^2$

$e^x = \tan\theta \qquad \ln e^x = \ln\tan\theta$

$\int \frac{\tan\theta}{\sec^2\theta}\,dx \qquad x = \ln\tan\theta$

$\int \tan\theta\cos^2\theta\,dx \qquad dx = \frac{\sec^2\theta}{\tan\theta}\,d\theta$

$\int \sin\theta\cos\theta\,dx$

$\int d\theta \qquad \theta = \tan^{-1}e^x$

$\theta + C$

$\tan^{-1}e^x + C$

~~d)~~ $\int \frac{2x+1}{x(x-1)^2}\,dx$

$\frac{2x+1}{x(x-1)^2} \equiv \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{(x-1)^2}$

$2x+1 = A(x-1)^2 + B(x^2-x) + (Cx+D)x$

$x=0 \Rightarrow A=1$

$x=1 \Rightarrow C+D=3$

$x=2 \Rightarrow 5 = 1+2B+(2C+D)2$

$4 = 2B+4C+2D$

$2 = B+2C+D$

$\underline{3 = C+D}$

$-1 = B+C$

7) a) $\lim\limits_{x \to 0} \left(\dfrac{3x^2 + 4x + 3}{5x^2 + 3x - 9} \right) \to \dfrac{-1}{3}$

b) $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\cos x}{x - \frac{\pi}{2}} = Y \to \dfrac{0}{0}$

$\lim\limits_{x \to \frac{\pi}{2}} \dfrac{-\sin x}{1} = -1$

$\cancel{c)}$ $\lim\limits_{x \to \infty} \left(1 + \dfrac{2}{x}\right)^x = \cancel{\lim\limits_{x \to \infty}} Y$

$\ln Y = x \ln\left(1 + \dfrac{2}{x}\right)$

$= x \ln \dfrac{x+2}{x}$

$= \ln \dfrac{xe^x + 2e^x}{x}$

$\to \ln \dfrac{xe^x + e^x + 2e^x}{1}$

$\dfrac{\ln}{e^x(x+3)}$

d) $\lim\limits_{x \to 0} \left(\dfrac{1}{\sin x} - \dfrac{1}{x} \right) = \lim\limits_{x \to 0} Y$

$Y = \dfrac{x - \sin x}{x \sin x}$

$\to \dfrac{1 - \cos x}{\sin x + x\cos x}$

$\to \dfrac{\sin x}{\cos x + \cos x - x\sin x} \to \dfrac{0}{2} = 0$

c) $\lim\limits_{x \to \infty} \left(1 + \dfrac{2}{x}\right)^x = \lim\limits_{x \to \infty} Y$

$Y = \left(1 + \dfrac{2}{x}\right)^x$

$\ln Y = x \ln\left(\dfrac{x+2}{x}\right)$

$= x \ln(x+2) - x \ln x$

$\cancel{Y = (x+2)^x - x^x}$

6) Given $Y = \sin x - \sin^{-1} x$

a) find $\frac{dY}{dX}$

$$\frac{dY}{dX} = \cos X - \frac{1}{\sqrt{1-X^2}}$$

b) $$\frac{d^2Y}{dX^2} = -\sin X - \frac{X}{(1-X^2)^{3/2}}$$

c) Show at $x=0$ there is a horizontal pt. of inflection $\left(\frac{dY}{dx} = \frac{d^2Y}{dX^2} = 0\right)$

$$\frac{dY}{dX} = \cos 0 - \frac{1}{\sqrt{1-0^2}} = 1 - 1 = 0$$

$$\frac{d^2Y}{dX^2} = -\sin 0 - \frac{0}{1} = 0 - 0 = 0$$

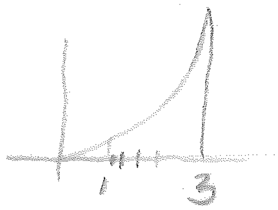
Pg 384

g) $$\lim_{x \to \frac{1}{2}} \frac{\ln(1-2X)}{\tan \pi X} \rightarrow \frac{-\infty}{\infty}$$

$$\rightarrow \frac{\frac{-2}{1-2X}}{\pi \sec^2 \pi X} \rightarrow \frac{\frac{0+4}{1-4X+4X^2}}{2\pi^2 \sec^2 \pi X \tan \pi X} =$$

Pg 426

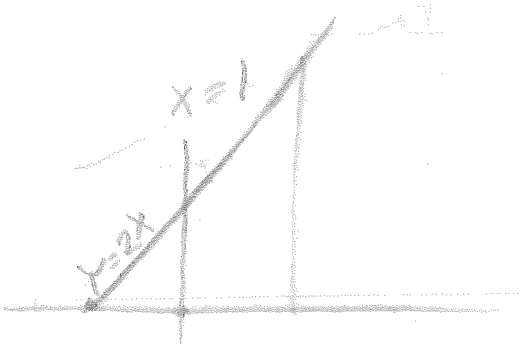
1) $\int_1^3 x^2 dx$



$\Delta x^3 + 2\Delta x^3 + 3\Delta x^3 \ldots n\Delta x^3$

$\Delta x^3(1+2+3\ldots n)$

$\Delta x\, y$



$\cancel{\Delta x(2\Delta x)+}$

$2\Delta x^2 + 2(2\Delta x^2) \ldots n(2\Delta x^2)$

$2\Delta x^2(1+2+3\ldots+n)$

$\Delta x = \frac{1}{n}$

$S_n = 2/n \quad \frac{\cancel{n}}{2}(1+n)$

$= \frac{1}{n}(1+n)$

$= \frac{1}{n} + \frac{n}{n}$

$\lim_{n\to\infty} S_n = 1$

Bob Marks
93

B156

Pg 260

9) a) $T = 2\pi\sqrt{\frac{L}{32}}$

$L = \frac{512}{9\pi^2}$

$\theta = A\sin\sqrt{\frac{32}{L}}\,T + B\cos\sqrt{\frac{32}{L}}\,t$

$\theta = \frac{\pi}{16} \Rightarrow t = 0$

$\theta = \frac{\pi}{16} = A\sin 0 + B\cos 0 = B$

$\theta = \frac{\pi}{16}\cos\sqrt{\frac{(32)(9\pi^2)}{512}}\,t$

$= \frac{\pi}{16}\cos\frac{3}{4}\pi t$

b) $\theta = B\cos\sqrt{\frac{32}{L}}\,t$

$\dot{\theta} = B\sqrt{\frac{32}{L}}\sin\sqrt{\frac{32}{L}}\,t$

$\dot{\theta}_{MAX} = \frac{\pi}{16}\left(\frac{3\pi}{4}\right)\sin\frac{\pi}{2}$

$\dot{\theta}_{MAX} = \frac{3\pi^2}{64}\sin\frac{\pi}{2}$

$\dot{\theta}_{MAX} = \frac{3}{64}\pi^2/SEC$

Pg 401

10 1) d) $Y = \int \frac{x+1}{(x-2)^2(x-3)^2} dx$

$$\frac{x-1}{(x-2)^2(x-3)^2} = \frac{A}{(x-2)} + \frac{B}{(x-2)^2} + \frac{C}{(x-3)} + \frac{D}{(x-3)^2}$$

$$x+1 = A(x-2)(x-3)^2 + B(x-3)^2 + C(x-2)^2(x-3) + D(x-2)^2$$

$A = 7,\ B = 3,\ C = -7,\ D = 4$

$$Y = 7\int \frac{dx}{x-2} + 3\int \frac{dx}{(x-2)^2} - 7\int \frac{dx}{(x-3)} + 4\int \frac{dx}{(x-3)^2}$$

$$Y = 7 \log\left(\frac{x-2}{x-3}\right) - \frac{3}{x-2} - \frac{4}{x-3} + C$$

Bob Marks
93

B156

Pg 260

9) a) $T = 2\pi\sqrt{\frac{L}{32}}$

$L = \frac{512}{9\pi^2}$

$\theta = A \sin\sqrt{\frac{32}{L}}\, T + B\cos\sqrt{\frac{32}{L}}\, t$

$\theta = \frac{\pi}{16} \Rightarrow t = 0$

$\theta = \frac{\pi}{16} = A\sin 0 + B\cos 0 = B$

$\theta = \frac{\pi}{16}\cos\sqrt{\frac{(32)(9\pi^2)}{512}}\, t$

$= \frac{\pi}{16}\cos\frac{3}{4}\pi t$

b) $\theta = B\cos\sqrt{\frac{32}{L}}\, t$

$\dot{\theta} = B\sqrt{\frac{32}{L}}\sin\sqrt{\frac{32}{L}}\, t$

$\dot{\theta}_{MAX} = \frac{\pi}{16}\left(\frac{3\pi}{4}\right)\sin\frac{\pi}{2}$

$\dot{\theta}_{MAX} = \frac{3\pi^2}{64}\sin\frac{\pi}{2}$

$\dot{\theta}_{MAX} = \frac{3}{64}\pi^2/\text{SEC}$

Pg 401

10 1) d) $Y = \int \frac{x+1}{(x-2)^2(x-3)^2}\,dx$

$$\frac{x-1}{(x-2)^2(x-3)^2} = \frac{A}{(x-2)} + \frac{B}{(x-2)^2} + \frac{C}{(x-3)} + \frac{D}{(x-3)^2}$$

$$x+1 = A(x-2)(x-3)^2 + B(x-3)^2 + C(x-2)^2(x-3) + D(x-$$

$$A = 7,\ B = 3,\ C = -7,\ D = 4$$

$$Y = 7\int \frac{dx}{x-2} + 3\int \frac{dx}{(x-2)^2} - 7\int \frac{dx}{(x-3)} + 4\int \frac{dx}{(x-3)^2}$$

$$Y = 7\log\left(\frac{x-2}{x-3}\right) - \frac{3}{x-2} - \frac{4}{x-3} + C$$

Pg 328

5) $F = M\frac{dV}{dt} = -KV$

$\frac{dV}{dt} = \frac{-K}{M}V \Rightarrow \frac{dt}{dV} = \frac{-M}{KV}$

$t = \frac{-M}{K}\ln V + C$

$t - C = \frac{1}{B}\ln V \Rightarrow \frac{M}{K} = \frac{1}{B}$

$B(T - C) = \ln V$

$V = \frac{e^{BC}}{e^{BT}}$

$V_0 = \frac{e^{BT}}{1} = e^{BC} \Rightarrow V = \frac{V_0}{e^{BT}}$

$\frac{dX}{dt} = V = \frac{V_0}{e^{BT}}$

$X = \frac{-V_0}{Be^{BT}} + C$

$t = 0 \Rightarrow X = 0 \Rightarrow C = \frac{V_0}{B}$

$X = \frac{V_0}{B}e^{-BT} + \frac{V^0}{B}$

$= \frac{V_0}{B}(1 - e^{-BT})$

6) $\frac{dV}{dt} = V\frac{dV}{dX}$

$-KV = V\frac{dV}{dX}$

$-K = \frac{dV}{dX}$

$-KX + C = V$

$V_0 - KX = V$

$V = V_0 - KX$

Pg 382

6) $X = \frac{\ln(1 + 100Kt)}{K}$

$\lim_{K \to 0} X = \lim_{K \to 0} \frac{\ln(1 + 100Kt)}{K} \to \frac{0}{0}$

$\to \frac{100t}{1 + 100Kt} \to 100t \Rightarrow \lim_{K \to 0} X = (100)t(= V_0 t)$

Pg 399

5) To use partial fraction integration, the numerator must be a degree less than the denominator. The proper way:

$$x^2-1 \overline{\big) \; x^2+1 \;} \quad 1$$
$$\underline{x^2-1}$$
$$2$$

10

$$Y = \int \frac{x^2+1}{x^2-1}\,dx = \int 1\,dx + \int \frac{2}{x^2-1}\,dx$$
$$= x + \int \frac{2}{x^2-1}$$
$$\frac{2}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$$
$$A(x+1) + B(x-1) = 2$$
$$x = -1 \Rightarrow B = -1$$
$$x = 1 \Rightarrow A = 1$$
$$Y = x + \ln(x-1) - \ln(x+1)$$
$$Y = x + \ln\left(\frac{x-1}{x+1}\right) + C$$

Pg 395

4) b) $Y = \int \cos^6 x\,dx$

$= \frac{1}{6}\cos^5 \sin x + \frac{5}{6}\int \cos^4 dx$

$= \frac{1}{6}\cos^5 \sin x + \frac{5}{24}\cos^3 x \sin x + \frac{15}{24}\int \cos^2 x\,dx$

$= \quad " \quad + \quad " \quad + \frac{15}{24}\left(\frac{1}{2}\cos x \sin x + \frac{1}{2}\int dx\right)$

$Y = \frac{1}{6}\cos^5 \sin x + \frac{5}{24}\cos^3 x \sin x + \frac{5}{16}\cos x \sin x + \frac{5}{16}x + C$ ✓

Pg 255

8) $Kd = 32m$

$K = 768$

$Y = A \sin\sqrt{\frac{K}{m}}\,t + B\cos\sqrt{\frac{K}{m}}\,t \qquad t=0,\ y=0 \qquad ?$

8

$\dot{Y} = A\sqrt{\frac{K}{m}}\,t \cos\sqrt{\frac{K}{m}}\,t - B\sqrt{\frac{K}{m}}\sin\sqrt{\frac{K}{m}}\,t$

$B = 0 \qquad A = \frac{1}{\sqrt{192}}$

$Y = \frac{1}{\sqrt{192}}\ \underset{\sin}{\cancel{\cos}}\sqrt{192}\,t$

9) $F = ma$

$m\ddot{x} = k(a - L - x) - k(a - l + x)$

$= -2kx$

$\ddot{x} = -\frac{2kx}{m}$

5

$x = A\sin\sqrt{\frac{2k}{m}}\,t + B\,\overset{\cos}{\cancel{\sin}}\sqrt{\frac{2k}{m}}\,t$

$x = D\cos\sqrt{\frac{2k}{m}}\,t$

10

Pg 311

10) $Y = (De^{kx})$

$\dot{Y} = k(De^{kx}) \Rightarrow \dot{Y} = kY$

Pg 362

6) a) $t = 2\pi\sqrt{\frac{L}{32}}$

$= 2\pi \frac{1}{2} \frac{1}{\sqrt{32L}} dL$

$= \frac{\pi}{\sqrt{32L}} \left(\frac{dL}{L}\right) L = \frac{\pi}{\sqrt{32}} (.01\sqrt{L})$

10

$= .01\, \pi \sqrt{\frac{L}{32}}$

$= .01\, \frac{t}{2}$

$\frac{dt}{t} = .005 = 5\%$

b) $\frac{dt}{t} = .005$

c) $(.005)(60)(24) = 7.2\,sec$

Bob Marks

100 $\underline{156}$

Pg 405

1) f) $Y = \int \frac{dx}{(x-1)^2(x^2+4x+5)}$

$$\frac{1}{(x-1)^2(x^2+4x+5)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+4x+5}$$

$$1 = A(x-1)(x^2+4x+5) + B(x^2+4x+5) + (Cx+D)(x-1)^2$$

$x = 1 \Rightarrow B = \frac{1}{10}$

$x = 2 \Rightarrow -\frac{12}{10} = 22A + C$

$x = 3 \Rightarrow -\frac{18}{5} = 72A + 12$

$$A = -\frac{3}{50},\quad B = \frac{1}{10},\quad C = \frac{3}{50},\quad D = \frac{10}{50}$$

$$Y = \int \frac{3}{50(x-1)} + \int \frac{1}{10(x-1)^2} + \int \frac{1}{50}\left(\frac{3x+10}{x^2+4x+5}\right)$$

$$Y = -\frac{3}{50}\ln(x-1) - \frac{1}{10(x-1)} + \frac{3}{100}\ln(x^2+4x+5) + \frac{4}{50}\tan^{-1}(x+2) + C$$

Pg 295

1) $r = 2.1 \times 10^7$; $r_1 = 3.15 \times 10^7$

$$\frac{r}{r_1} = \frac{2}{3}$$

$$T = \sqrt{\frac{2}{1.4 \times 10^{16}}}\, r_1^{\frac{3}{2}}\left(\frac{1}{2}\sin^{-1}\sqrt{\frac{r}{r_1}} - \frac{1}{2}\sqrt{\frac{r}{r_1}}\sqrt{1-\frac{r^2}{r_1^2}} - \frac{\pi}{4}\right)$$

$$= \sqrt{\frac{2}{1.4 \times 10^{16}}}\,(31.6 \times 10^6)^{\frac{3}{2}}\left(\frac{1}{2}\sin^{-1}\sqrt{\frac{2}{3}} - \frac{1}{2}\sqrt{\frac{2}{3}}\sqrt{\frac{5}{9}} - .78\right)$$

$$= (2.14 \times 10^3)(.58)$$

$T = 1280$ sec

Pg 411

1) b) $Y = \int \frac{dx}{1+x+x^2}$

$= \frac{2}{\sqrt{3}} \tan^{-1} \frac{2x+1}{\sqrt{3}} + C$

10

f) $Y = \int \frac{dx}{x\sqrt{x^2+9}}$

$= \frac{1}{3} \ln \left(\frac{\sqrt{x^2+9} - 3}{x} \right) + C$

Pg 319

7) $\frac{ds}{dt} = -ks$

$\frac{dt}{ds} = \frac{-1}{ks}$ $\left(T = \frac{1}{k} \ln s + C \right.$

$S = De^{-kt}$

$T = 0 \Rightarrow S = 200 \Rightarrow D = 200$

$S = 200e^{-kt}$

10

$S = 100 \Rightarrow t = 2$

$\frac{1}{2} = e^{-2K} \Rightarrow K = .347$ if $S = 50$

$50 = 200e^{-.347T}$

$\frac{1}{4} = e^{-.347T}$

$T = \frac{\ln 1 - \ln 4}{-.347}$

$= 4$ min

Pg 406

1) b) $Y = \int \frac{2x}{(1+x)(1+x^2)^2}\,dx$

$$\frac{2x}{(1+x)(1+x^2)^2} = \frac{A}{1+x} + \frac{Bx+C}{1+x^2} + \frac{Dx+E}{(1+x^2)^2}$$

$$2x = A(1+x^2)^2 + (Bx+C)(1+x)(1+x^2) + (Dx+E)(1+x)$$

$$x = -1 \Rightarrow A = \frac{-1}{2}$$

$$x = j \Rightarrow 2j = (Dj+E)(1+j)$$

$$2j = Dj - D + E + Ej$$

$$2j = Dj + Ej \qquad 0 = E - D$$

$$E + D = 2 \qquad E = D$$

$$D = E = 1$$

$$x = 0 \Rightarrow 0 = A + C + E$$

$$C = \frac{-1}{2}$$

$$x = 1 \Rightarrow 2 = 4A + 4(B+C) + 2(D+E)$$

$$1 = 2A + 2B + 2C + D + E$$

$$\cancel{1} = -\cancel{1} + 2B + \cancel{1} + \cancel{1} + 1$$

$$B = \frac{1}{2}$$

$$Y = \frac{1}{2}\int \frac{-1}{1+x} + \frac{1}{2}\int \frac{x-1}{x^2+1} + \int \frac{x+1}{(1+x^2)^2}$$

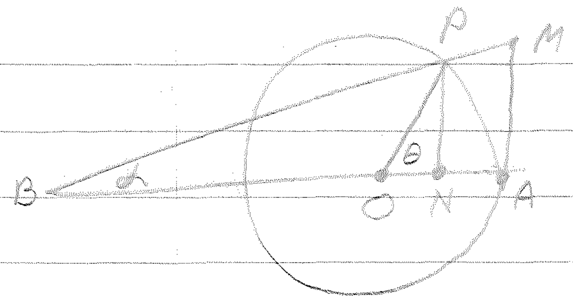
$$= -\frac{1}{2}\ln(1+x) + \frac{1}{4}\ln(1+x^2) - \frac{1}{2}\tan^{-1}x + \int \frac{x+1}{(1+x^2)^2}$$

$$= \frac{1}{2}\ln\frac{\sqrt{1+x^2}}{x-1} - \frac{1}{2(1+x^2)} + \frac{x}{2(1+x^2)}$$

$$= \frac{1}{2}\ln\frac{\sqrt{1+x^2}}{x-1} + \frac{(x-1)}{2(1+x^2)} + c$$

Pg 387

6) $\triangle BNP \sim \triangle BAM$



$\frac{PN}{BN} = \frac{MA}{BA}$

$PN = r\sin\theta$

$ON = r\cos\theta$

$MA = BM\sin\alpha$

$BA = BM\cos\alpha$

$\frac{r\sin\theta}{BO - r\cos\theta} = \frac{\sin\alpha}{\cos\alpha}$

$\overline{BO}\sin\alpha - r\cos\theta\sin\alpha = r\sin\theta\cos\alpha$

10 $\overline{BO} = \frac{r(\cos\theta\sin\alpha + \sin\theta\cos\alpha)}{\sin\alpha}$

$Y = \lim_{P\to A}\overline{BO} = \lim_{P\to A}\frac{r(\cos\theta\sin\alpha + \sin\theta\cos\alpha)}{\sin\alpha}$

$Y = \lim_{P\to A}\frac{r\sin(\theta+\alpha)}{\sin\alpha} \to \frac{r\sin 2\alpha}{\sin\alpha} \to \frac{2r\cos}{\cos}$ $\alpha\to 0$

$2r$

$\lim_{P\to A}\overline{BO} = 2r$

Pg 409

1) b) $\int \frac{dx}{1+\sin x + \cos x}$

$x = \tan^{-1} u$

$\frac{du}{dx} = \frac{2}{1+u^2}$

$\cos x = \frac{1-u^2}{1+u^2}, \quad u = \sqrt{\frac{1-\cos x}{1+\cos x}}$

$\int \frac{dx}{1+\sin x+\cos x} = \int \frac{dx}{1+\frac{2u}{u^2+1}+\frac{1-u^2}{1+u^2}} = \int \frac{2du}{2u+2}$

$= \log(u+1) + C = \ln\left(1+\sqrt{\frac{1-\cos x}{1+\cos x}}\right) + C$

$= \log\left(1+\frac{\sin x}{1+\cos x}\right) + C$

c) $\int \frac{dx}{\sin x + \tan x}$

$x = 2\tan^{-1} u \qquad u = \sqrt{\frac{1-\cos x}{1+\cos x}}$

$dx = \frac{2}{1+u^2}\,du \qquad u = \tan\frac{x}{2}$

$\int \frac{2du/1+u^2}{\frac{2u}{1+u^2}+\frac{2u}{1-u^2}} = \int \frac{2du/1+u^2}{\frac{2u-2u^3+2u+2u^3}{1+u^4}} =$

$\int \frac{2du}{\frac{4u}{1-u^2}} = \frac{1}{2}\int\frac{du}{u} - \frac{1}{2}\int u\,du$

$= \frac{1}{2}\ln u - \frac{u^2}{4} =$

$\frac{1}{2}\log\tan\frac{x}{2} - \frac{1}{4}\tan^2\frac{x}{2} + C$

Pg 320

13) $q(t)$ = lbs salt at t

$t = 0 \Rightarrow q = 150$

$\frac{dq}{dt} = \frac{1}{50}(100-q)$

$\frac{dt}{dq} = \frac{50}{100-q} = 50 \cdot \frac{1}{100-q}$

$t = 50 \ln(100-q) + C$

$\frac{t-c}{50} = \ln(q-100)$

10 $e^{\frac{t-c}{50}} = q-100$

$q = 100 + C' e^{\frac{t}{50}}$

$C' = e^{\frac{c}{50}}$

$150 = 100 + C' e^{0}$

$C' = 50$

$Q = 100 + 50e^{\frac{-60}{50}}$ $(t = 60)$

$Q = 100 + 50e^{-6/5} = 100 + 50e^{-1.2}$

$= 100 + 50(.302) = 115.1$

Pg 337

3) $\alpha = \tan^{-1} \sqrt{\frac{k}{32}}\, 1000$

$C = \frac{-1}{k} \ln \cos \alpha$

10 $Y = \frac{-1}{k} \ln \cos \tan^{-1} \sqrt{\frac{k}{32}}\, 1000$

$Y = \frac{-1}{k} \ln \frac{\sqrt{32}}{\sqrt{32 + 1000^2 k}}$

$y = \frac{1}{2k} \ln \left[1 + \frac{k 10^6}{32}\right]$

Pg 399
§

3) $\frac{dx}{dt} = k(a-x)(b-x)$

$\frac{dt}{dx} = \frac{1}{k(a-x)(b-x)}$

$t = \frac{1}{k}\int \frac{1}{(a-x)(b-x)}\,dx$

$= \frac{1}{k}\int \frac{A}{a-x} + \frac{B}{b-x}$

$1 = A(b-x) + B(a-x)$

$B = \frac{1}{a-b} \qquad A = \frac{1}{b-a}$

$\frac{1}{k}\int \frac{\frac{1}{b-a}}{a-x}\,dx + \frac{1}{k}\int \frac{\frac{1}{a-b}}{b-x}\,dx$

$t = \frac{-1}{k(b-a)}\ln(a-x) + \frac{1}{k(b-a)}\ln(b-x) + C$

10 $t = \frac{-1}{k(b-a)}\ln\frac{a-x}{b-x} + C$

$= \frac{-1}{k(b-a)}\ln C_1\frac{(a-x)}{(b-x)}$

$\frac{a-x}{b-x} = C_2 e^{-k(b-a)k}$

$t = 0 \Rightarrow x = 0$

$C_2 = \frac{a}{b}$

$\frac{a-x}{b-x} = \frac{a}{b}\left(e^{-k(b-a)t}\right) / \; ba - bx = abe^{-k(b-a)t} - axe^{-k(b-a)t}$

$x(ae^{-k(b-a)t} - b) = abe^{-k(b-a)t} - ab$

$x = \frac{ab(e^{k(b-a)t} - 1)}{ae^{k(b-a)t} - b}$

... TIVES OF INVERSE TRIG FUNCTIONS

1. A MAN IS WALKING AT THE RATE OF 4 MILES/HR TOWARD THE FOOT OF A TOWER 60 FT. HIGH STANDING ON LEVEL GROUND. AT WHAT RATE IS THE ANGLE OF ELEVATION OF THE TOP CHANGING WHEN HE IS 80 FT FROM THE FOOT OF THE TOWER? (2.112 RAD/MIN — INCREASING)

2. A SEARCHLIGHT, LOCATED 100 FT FROM A STRAIGHT ROAD, IS TRAINED UPON A CAR RUNNING ALONG THE ROAD AT 30 MILES PER HR; AT WHAT RATE PER MINUTE MUST THE LIGHT BE ROTATING WHEN THE CAR 200 FT FROM THE NEAREST POINT OF THE ROAD TO THE LIGHT?

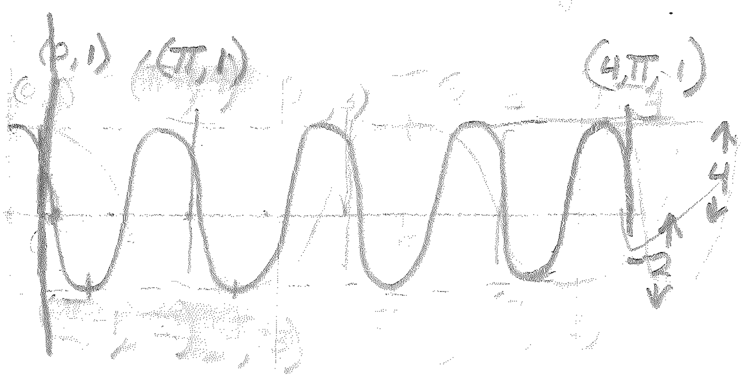
3. A SIGN BOARD 10 FT HIGH IS ERECTED WITH ITS LOWER EDGE 13 FT ABOVE THE GROUND. AT WHAT DISTANCE WOULD A MAN WHOSE EYES ARE 5 FT ABOVE THE GROUND OBTAIN THE CLEAREST VIEW OF THE SIGN? (12 FT)

4. AN AIRPLANE 1 MILE HIGH IS FLYING HORIZONTALLY WITH A VELOCITY OF 100 MILES PER HOUR, DIRECTLY AWAY FROM THE OBSERVER. AT WHAT RATE IS THE ANGLE OF ELEVATION OF THE AIRPLANE CHANGING WHEN THE POINT DIRECTLY UNDER THE AIRPLANE IS $\frac{1}{5}$ MILE FROM THE OBSERVER?

5. A KITE IS 60 FT HIGH, WITH 100 FT OF CORD OUT. IF THE KITE IS MOVING HORIZONTALLY 4 MILES PER HOUR DIRECTLY AWAY FROM THE BOY WHO IS FLYING IT, FIND THE RATE OF CHANGE OF THE ANGLE OF ELEVATION OF THE KITE STRING, ASSUMING THE KITE STRING TO BE IN A STRAIGHT LINE. (2.112 RAD/MIN — DECREASING)

MARKS

9 / 5



... DONE IN THE TEXT & IN CLASS, STATE WHAT THE FOLLOWING TWO LIMITS ARE

(a) $\lim_{x \to 0} \frac{\sin x}{x} = 1$ (b) $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$

USING THESE LIMITS AND HONEST TRICKERY, DETERMINE THE FOLLOWING

(a) $\lim_{x \to 0} \frac{1 - \cos 2x}{x} = 0$

(b) $\lim_{x \to 0} \frac{1 - \cos x}{\sin x} = 0$

FIND THE DERIVED FUNCTION OF (I.E., DIFFERENTIATE) EACH OF THE FOLLOWING FUNCTIONS

(a) $y = \cos(3x + 2)$

$y' = -3\sin(3x+2)$

(b) $y = \sec\sqrt{z}$

$U = \sqrt{z} \quad Y = \sec U$

$\frac{1}{2\sqrt{z}} \sec\sqrt{z} \tan\sqrt{z}$

(c) $y = \sin(\tan x)$

$U = \tan x \quad Y = \sin U$

$\frac{dU}{dx} = \sec^2 x \quad \frac{dY}{dU} = \cos U$

$\frac{dY}{dx} = \sec^2 x \cos(\tan x)$

(d) $y = \sqrt{2 - \cos^2 x}$

$U = \cos^2 x \qquad Y = (2 - U)^{\frac{1}{2}}$

$\frac{dU}{dx} = -2\cos x \sin x \qquad V = 2 - U \quad Y = V^{\frac{1}{2}}$

$\frac{dV}{dU} = -1 \quad \cancel{\frac{dY}{dV} = \frac{1}{2} V^{-\frac{1}{2}}}$

$\frac{dY}{dx} = (+2\cos x \sin x)\frac{1}{2}(2-U)^{-\frac{1}{2}} = \cos x \sin x (2 - \cos^2 x)^{-\frac{1}{2}}$

156 Boo Marks

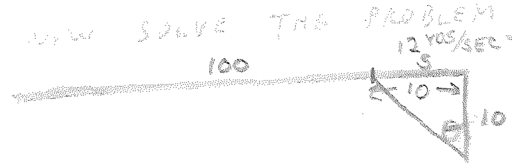
... HOW FAST IS THE TV CAMERA TURNING ... (IN RAD/SEC)? DO THE FOLLOWING:

(1) SKETCH THE SITUATION & LABLE THE PERTINENT QUANTITIES

(2) STATE WHAT IS GIVEN & WHAT IS REQUIRED. $\frac{ds}{dt} = 12$ YDS/SEC

(3) STATE WHAT IS NEEDED AS A FUNCTION OF WHAT. $\frac{d\theta}{dt} = \frac{ds}{dt}\frac{d\theta}{ds}$

(4) NOW SOLVE THE PROBLEM



$S = 10\tan\theta \qquad \theta = \tan^{-1}\frac{S}{10}$

$\frac{ds}{d\theta} = 10\sec^2\theta$

$\frac{d\theta}{ds} = \frac{1}{10}\cos^2\theta$

$= \frac{1}{10}\cos^2\left(\tan^{-1}\frac{S}{10}\right)$

$\frac{d\theta}{dt} = \frac{1}{10}\cos^2\left(\tan^{-1}\frac{5}{10}\right)12$

$\frac{12}{10}\cos^2\left(\frac{\pi}{4}\right)$

$\frac{1}{2}\left(\frac{6}{5}\right) = \frac{6}{10} = \frac{3}{5}$ rad/sec

B. GIVEN THE FOLLOWING DERIVED FUNCTIONS, FIND y:

(a) $y' = \tan^3 2x \sec^2 2x \qquad U = 2x \qquad \dot{Y} = \tan^3 U \sec^2 U$

$V = \tan U$

$\dot{Y} = V^3 \frac{dV}{dU}$

$Y = \frac{1}{8}\tan^4 2X + C$

(b) $y' = (1 + \sin^2 x)\sin x \cos x \qquad U = \sin X \qquad \dot{Y} = (1 + U^2)U\frac{du}{dx}$

$= (U + U^3)\frac{du}{dx}$

$= \frac{1}{2}U^2 + \frac{1}{4}U^4 + C = \frac{1}{2}\sin^2 X + \frac{1}{4}\sin^4 X + C$

(c) $y' = \tan^2 x \quad = \tan^2 X(\sec X)^2$

$U = \tan X \qquad V = \sec X$

$\frac{du}{dx} = \sec^2 X \qquad \frac{dv}{dx} = \sec X \tan X$

$\dot{Y} =$

(d) $y' = \frac{\sec^2 x}{(1 + \tan x)^2} = \sec^2 X(1 + \tan X)^{-2} \qquad U = \tan X$

$= \frac{-1}{(1 + \tan X)} \qquad y = -\frac{1}{(1 + \tan x)}$

BOB MARKS
8F17 C-3

~~90~~

90 %

don't
write in red ink

good
work

1) $\dot{y} = \tan^2 x = \sec^2 x - 1$

$y = \tan x - x + C$ ✓

2) $\dot{y} = \tan^4 x = \cancel{\sec^2 x \sec^2 x} \tan^2 x \tan^2 x$

$= \cancel{(\sec^2 x - 1)(\sec^2 x - 1)} (\sec^2 x - 1)^2$

$\cancel{y \; \frac{dy}{dx} = \sec^2 x - (\tan^2 x - x)^2}$

↓
TO BOTTOM

$\cancel{y = \tan^4 x - 2x\tan^2 x + x^2 + C}$

3) $x = \tan\left(\frac{\pi}{4} - \frac{t}{2}\right)$

$u = \frac{\pi}{4} - \frac{t}{2}$ $\quad x = \tan u$

$\frac{du}{dt} = -\frac{1}{2}$ $\quad \frac{dx}{du} = \sec^2 u$

$\frac{dx}{dt} = -\frac{1}{2}\sec^2\left(\frac{\pi}{4} - \frac{t}{2}\right)$ ✓

4) $y = \sec\sqrt{x}$

$u = \sqrt{x}$ $\quad y = \sec u$

$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$ $\quad \frac{dy}{du} = \sec u \tan u$

$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} \sec\sqrt{x} \tan\sqrt{x}$ ✓

2) $u = \sec^2 x - 1$ $\quad \dot{y} = u^2$

$\frac{du}{dx} = 2\sec x \tan x - x$ $\quad \frac{1}{3}u^3$

$\left(\frac{1}{3}\right) \frac{1}{2\sec x \tan x - x} (\sec^2 - 1)^3$

$y = \sec\sqrt{x}$

$y = \sec\sqrt{x} \tan\sqrt{x} \left(\frac{1}{2\sqrt{x}}\right)$

neat

5) $\tan^4 x = \tan^2 x\,(\sec^2 x - 1)$

$= \tan^2 x \sec^2 x - \tan^2 x$

$= \tan^2 x \sec^2 x - \sec^2 x + 1$

$y = \frac{1}{3}\tan^3 x - \tan x + x + C$

Name: Robert Marks
Box # 156 Seat # 5-A 80

10 1. $y = \frac{5(x+1)^2(x+2)}{(x+3)(x+4)^2}$ Find values of y and y' when $x = 2$.

$= \frac{\cancel{5}(3)^2(4)}{(\cancel{5})(6)^2} = \frac{36}{36} = 1$ ✓

$Y = 5(x-1)^2(x+2)(x+3)^{-1}(x+4)^{-2}$

$Y' = \frac{5(2)(x-1)(x+2)}{(x+3)(x+4)^2} + \frac{5(x-1)^2}{(x+3)(x-4)^2} - \frac{5(x-1)^2(x+1)}{(x+3)^2(x+4)^2} - \frac{10(x-1)^2(x+2)}{(x+3)(x+4)^3}$

$= \frac{(10)(1)(2)}{(5)(36)} \frac{1}{6} + \frac{5}{(5)(4)} \frac{1}{4} - \frac{(1)}{15(12)} - \frac{2(1)4}{(2)(3)^3} = \frac{1}{6} + \frac{1}{4} - \frac{1}{60} - \frac{4}{27}$

$\frac{90 + 135 - 8 - 80}{540}$

$\frac{68}{270} = \frac{34}{135}$

10 2. $Y = \int (\tan x + \cot x)^2\, dx$

$\int \left(\tan x + \frac{1}{\tan x}\right)^2 dx$

$\int \left(\frac{\tan^2 x + 1}{\tan x}\right)^2 dx$

$\int \left(\frac{\sec^2 x}{\tan x}\right)^2 dx$

$Y = \int \frac{\sec^4 x}{\tan^2 x}$

$= \int \sec^2 x - \sec^4 x$

$= \int \sec^2 x - (\sec^2 x(1 + \tan^2 x))$

$U = \tan x \quad dx = \sec^2 du$

$p = (1 + U^2)\, du$

$U - \frac{1}{3}U^3$

$Y = \tan x - \tan x + \frac{1}{3}\tan^3 x = \frac{1}{3}\tan^3 x$

0 3. Find $\int \sec x\, dx$ by first showing $\sec x = \frac{1}{2}\cos x \left[\frac{1}{1+\sin x} + \frac{1}{1-\sin x}\right]$

$Y = \int \sec x\, dx$

$= \int \frac{1}{2}\cos x\left(\frac{1}{1+\sin x} + \frac{1}{1-\sin x}\right)$

$= \int \frac{1}{2}\cos x\left(\frac{2}{\cos^2 x}\right)$

$= \int \cos x$

$= \sin x$

very bad algebra and trig!

$\cos x = \frac{1}{\sec x}$ what!

$\sec x = \frac{1}{2}\cos x(2)$

$\sec x = \frac{1}{2}\cos x(1 + \csc x + 1 + \csc x)$

$\sec x = \frac{1}{2}\cos x\left(\frac{1}{1+\sin x} + \frac{1}{1-\sin x}\right)$

$1 + \csc x \neq \frac{1}{1+\sin x}$

$1 + \csc x = 1 + \frac{1}{\sin x} = \frac{1+\sin x}{\sin x}$

P.2 7 Mar 66.

20 4. $\int \frac{7x+6}{x^3+x^2-6x}\,dx = \int \frac{7x+6\;dx}{x(x^2+x-6)} = \int \frac{7x+6}{x(x+3)(x-2)}$

$$\frac{7x+6}{x(x+3)(x-2)} = \frac{A}{x} + \frac{B}{x+3} + \frac{C}{x-2}$$

$$7x+6 = A(x+3)(x-2) + B(x)(x-2) + C(x+3)x$$

$x=0 \Rightarrow A=-1$

$x=-3 \Rightarrow B=-1$

$x=2 \Rightarrow C=2$

$$\int \frac{2}{x-2} - \frac{1}{x+3} - \frac{1}{x}$$

$$= 2\log(x-2) - \log(x+3) - \log x$$

$$= \log\frac{(x-2)^2}{x(x+3)} + C$$

20 5. (a) $\lim_{x\to\frac{\pi}{2}} (\pi-2x)\tan x = \lim_{x\to\frac{\pi}{2}} Y$

$Y = \frac{\pi-2x}{\operatorname{ctn} x} \to \frac{0}{0}$

$\to \frac{+2}{+\csc^2 x} = 2\sin^2 x \to 2$

$\lim_{x\to\frac{\pi}{2}} Y = 2$

(b) $\lim_{x\to\frac{\pi}{2}} \sec 3x \cos 5x = \lim_{x\to\frac{\pi}{2}} Y$

$= \frac{\cos 5x}{\cos 3x} \to \frac{0}{0}$

$\to \frac{5\sin 5x}{3\sin 3x} \Rightarrow \frac{5}{-3}$

$\lim_{x\to\frac{\pi}{2}} Y = \frac{-5}{3}$

20 6. $Y = \int 27x^2 e^{3x}\,dx$

$u = 27x^2 \quad dv = e^{3x}$

$du = 54x\,dx \quad v = \frac{1}{3}e^{3x}$

$Y = 9x^2e^{3x} - \int 18x e^{3x}\,dx$

$u = 18x \quad dv = e^{3x}$

$du = 18\,dx \quad v = \frac{1}{3}e^{3x}$

$Y = 9x^2e^{3x} - (6xe^{3x} - \int 6e^{3x})$

$Y = 9x^2e^{3x} - 6xe^{3x} + 2e^{3x} + C$

R 69.0206

OK Make-up for Quiz
Fri 31 Jan

1) $\dot{Y} = \sec^3 X \tan X$

$U = \sec X$

$\frac{du}{dx} = \sec X \tan X$

$\dot{Y} = U^2 \frac{du}{dx}$

$Y = \frac{1}{3} \sec^3 X + C$

2) $\dot{Y} = \tan^3 2X \sec^2 2X$

$U = \tan 2X$

$\frac{du}{dx} = 2 \sec^2 2X$

$\dot{Y} = \frac{1}{2} U^3 \frac{du}{dx}$

$Y = \frac{1}{8} \tan^4 2X + C$

3) $\dot{Y} = \sqrt{\cot^3 X}\ \csc^2 X$

$= \cot^{\frac{3}{2}} X \csc^2 X$

$U = \cot X$

$\frac{du}{dx} = -\csc^2 X$

$\dot{Y} = -U^{\frac{3}{2}} \frac{du}{dx}$

$Y = -\frac{2}{5} \cot^{\frac{5}{2}} X + C$

4) $\dot{Y} = \sec^5 X \tan X$

$U = \sec X$

$\frac{du}{dx} = \sec X \tan X$

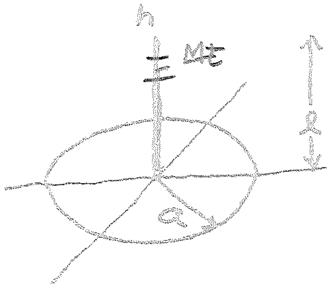
$\dot{Y} = U^4 \frac{du}{dx}$

$Y = \frac{1}{5} \sec^5 X + C$

15F25 April 1969 Calculus III Name Bob Marks BoxNo. 156

3. Find the attraction of a solid disc of mass Mt per unit of surface area on a rod of uniformly distributed mass m and length ℓ normal to the ~~xxx~~ center of the disc. Note: The attraction of the disc for a point mass is

$$2\pi G m M t\left(1 - \frac{h}{\sqrt{a^2 + h^2}}\right)$$



$$dF = 2\pi G M t\left(\frac{m}{\ell}dh\right)\left(1 - \frac{h}{\sqrt{a^2+h^2}}\right)$$

-25

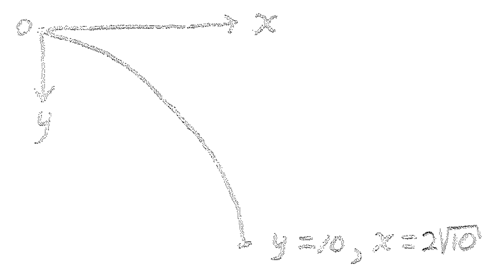
$$F = \frac{2\pi G mMt}{\ell}\int_0^\ell \left(1 - \frac{h}{\sqrt{a^2+h^2}}\right) dh$$

=

=

$$= \frac{2\pi G mMt}{\ell}\left[\ell + a - \sqrt{\ell^2+a^2}\right]$$

4. If a particle slides down the curve of $y = x^2/4$ and it is given a shove to start it at an initial velocity of 8 feet/second, find the time it takes to go from $y = 0$ to $y = 10$.



$$t = \int_{y=y_0}^{y=y_t} \frac{ds}{\sqrt{64(y-y_0)+V_0^2}}$$

$$Y = \frac{x^2}{4}$$

$$\frac{dY}{dx} = \frac{x}{2}$$

$$S = \int \sqrt{1 + \left(\frac{dY}{dx}\right)^2}\, dx$$

$$dS = \sqrt{1 + \frac{x^2}{4}}\, dx$$

$$x = 2Y^{\frac{1}{2}}$$

$$dx = Y^{-\frac{1}{2}} dY$$

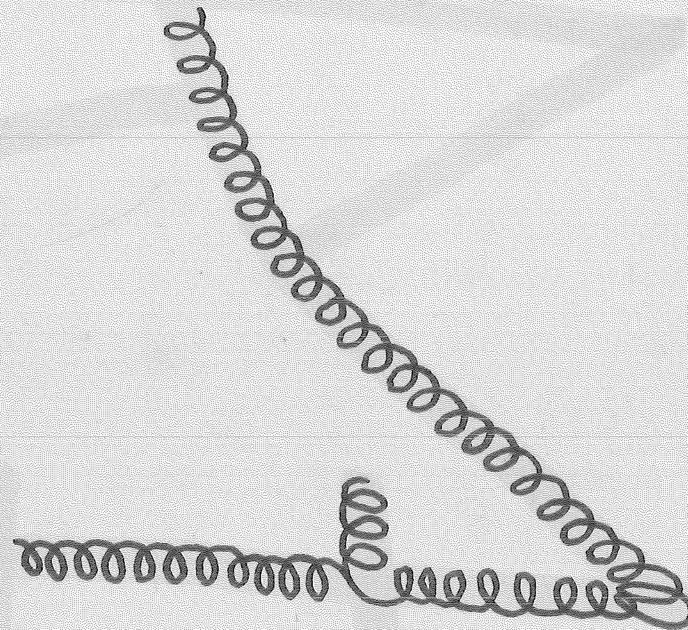
$$t = \int_0^{10} \frac{\sqrt{1 + \frac{x^2}{4}}\, dY}{Y^{\frac{1}{2}}\sqrt{64Y + 64}}$$

$$= \int_0^{10} \frac{\cancel{(1+Y)^{\frac{1}{2}}}}{8Y^{\frac{1}{2}}\cancel{(Y+1)^{\frac{1}{2}}}} dY$$

$$= \int_0^{10} \frac{1}{8Y^{\frac{1}{2}}} dY$$

$$t = \left.\frac{Y^{\frac{1}{2}}}{4}\right|_0^{10}$$

$$= \frac{\sqrt{10}}{4} \text{ SEC}$$



I) HYPERBOLIC FUNCTIONS

A) IDENTITIES

1) $\cosh x = \frac{e^x + e^{-x}}{2}$

2) $\sinh x = \frac{e^x - e^{-x}}{2}$

3) $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x + e^{-x}}{e^x + e^{-x}}$

4) $\coth x = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

5) $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$

6) $\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$

7) $\cosh^2 x = \sinh^2 x = 1$

8) $\sinh x = -\sinh(-x)$

9) $\sinh 2x = 2 \sinh x \cosh x$

B) DIFFERENTIATION OF HYPERBOLIC FUNCTIONS

1) $\frac{d}{dx} \sinh x = \cosh x$

2) $\frac{d}{dx} \cosh x = \sinh x$

II) VECTOR MANIPULATION IN 3D

A) DOT PRODUCT ($\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$, $b = b_1\vec{i} + b_2\vec{j} + b_3 k$, etc.)

1) $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos(\vec{a}, \vec{b})$

2) $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

B) CROSS PRODUCT

1) $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin(\vec{a}, \vec{b})$

2)
$$\vec{a} \times \vec{b} = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

C) COLINEAR PTS.

$A = (a_1, a_2, a_3)$ $B = (b_1, b_2, b_3)$ $C = (c_1, c_2\ c_3)$

A, B, AND C ARE COLINEAR IFF

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = 0$$

D) // AND $\perp$ VECTORS

1) $\vec{a} \perp \vec{b}$ IFF $\vec{a} \cdot \vec{b} = 0$ $(\cos 90° = 0)$

2) $\vec{a} \parallel \vec{b}$ IFF $\vec{a} \times \vec{b} = 0$ $(\sin 0° = 0)$

3) GIVEN TWO LINES WITH POINTS:

$L_1 \{(a_1, a_2, a_3)(b_1 b_2 b_3)\}$ $L_2 \{(c_1, c_2, c_3)(d_1, d_2, d_3)\}$

a) $L_1 \perp L_2$ IFF

$$(a_1 - b_1)(c_1 - d_1) + (a_2 - b_2)(c_2 - d_2) + (a_3 - b_3)(c_3 - d_3) = 0$$

b) $L_1 \parallel L_2$ IFF

$$\frac{a_1 - b_1}{c_1 - d_1} = \frac{a_2 - b_2}{c_2 - d_2} = \frac{a_3 - b_3}{c_3 - d_3}$$

E) LINE EQUATION FROM 2 POINTS

$A = (a_1, a_2 a_3)$ $B = (b_1 b_2 b_3)$

1) $X = a_1 + (b_1 - a_1)t$

$Y = a_2 + (b_2 - a_2)t$

$Z = a_3 + (b_3 - a_3)t$

2) $$\frac{X - a_1}{b_1 - a_1} = \frac{Y - a_2}{b_2 - a_2} = \frac{Z - a_3}{b_3 - a_3}$$

3) IF $a_1 = b_1$, $X = a_1$

$a_2 = b_2$, $Y = a_2$

$a_3 = b_3$, $Y = a_3$

F) PLANE PASSING THRU A POINT $\perp$ TO A LINE

$L_{\text{DIRECTIONAL}} = (A, B, C)$ $P_0 = (X_1 Y_1 Z_1)$

$$A(X - X_1) + B(Y - Y_1) + C(Z - Z_1) = 0$$

G) EQUATION OF A PLANE GIVEN 3 POINTS

$(a_1 a_2 a_3)$ $(b_1 b_2 b_3)$ $(c_1 c_2 c_3)$

1) $a_1 A + b_1 B + c_1 C = 1$

$a_2 A + b_2 B + c_2 C = 1$

$a_3 A + a_3 B + c_3 C = 1$

2) SOLVE FOR A, B, C

3) $AX + BY + CZ = 1$

III) CONTINUITY (DEF)

$w = f(x, y)$ IS CONTINUOUS AT (x_0, y_0)

IFF $w \to w_0 (= f(x_0, y_0))$ AS $(x, y) \to (x_0, y_0)$

IV) NORMALS AND TANGENTS by PARTIAL DIFFER.

$w = f(x, y)$ AT (x_0, y_0, z_0)

$A = \frac{\partial w}{\partial x} \quad B = \frac{\partial w}{\partial y}$

A) TANGENT PLANE

1) $w - w_0 = \frac{\partial w}{\partial x}(x - x_0) + \frac{\partial w}{\partial y}(y - y_0)$

2) $w - w_0 = A(x - x_0) + B(y - y_0)$

B) NORMAL LINE

1) $x = x_0 + At$

$y = y_0 + Bt$

$w = w_0 - t$

2) $\frac{x - x_0}{A} = \frac{y - y_0}{B} = \frac{w - w_0}{-1}$

V) THE GRADIENT (∇w) AT $P_0 (x_0, y_0, z_0)$

AND $w = f(x, y, z)$

A) $\frac{\partial w}{\partial x}\hat{i} + \frac{\partial w}{\partial y}\hat{j} + \frac{\partial w}{\partial z}\hat{k} = \nabla w$

B) ∇ (= DEL OPERATOR) $= \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$

C) DIRECTIONAL DIRIVITIVE

1) $\frac{dw}{ds} = \frac{\partial w}{\partial x}\cos\alpha + \frac{\partial w}{\partial y}\cos\beta + \frac{\partial w}{\partial z}\cos\gamma$

2) $\frac{dx}{ds} = \cos\alpha$

$\frac{dy}{ds} = \cos\beta$

$\frac{dz}{ds} = \cos\gamma$

3) AT PT. $P_0(x_0, y_0, z_0)$ IN DIRECTION OF

$\vec{l} = A\hat{i} + B\hat{j} + C\hat{k}$

DIREC. DIRIV. $= \frac{(x_0\hat{i} + y_0\hat{j} + z_0\hat{k}) \cdot (A\hat{i} + B\hat{j} + C\hat{k})}{\sqrt{A^2 + B^2 + C^2}}$

VI) EXACT DIFFERENTIAL

A) $z = f_x(x,y)\,dx + f_y(x,y)\,dy$

IS AN EXACT DIFF. IF

IF $\frac{\partial f_x(x,y)}{\partial x} = \frac{\partial f_y(x,y)}{\partial y}$

VII) INFINITE SERIES (CONVERGENCE & SUCH)

$S_n = a_1 + a_2 + a_3 + a_4 \ldots + a_n$ → IF SEQUENCE OF PARTIAL SUMS HAS A LIMIT

A) $\sum_{n=1}^{\infty} a_n \left(= \lim_{n\to\infty} S_n\right)$ CONVERGES IFF $\lim_{n\to\infty} a_n = 0$

B) IF $\sum b_n$ IS CONVERGENT, AND $0 < a_n < b_n$, THEN $\sum a_n$ CONVERGES

C) INTEGRAL TEST

$\sum_{n=1}^{\infty} a_n$ IS CONVER. IFF $\int_1^{\infty} f(x)$ IS FINITE WHERE $f(x) = a_n$

1) $\sum_{n=1}^{\infty} a_n > \int_1^{\infty} f(x)\,dx$

2) $\sum_{n=2}^{\infty} a_n < \int_1^{\infty} f(x)\,dx$

3) $\sum_{n=1}^{\infty} a_n < \int_0^{\infty} f(x)\,dx$

D) IF $0 < \frac{a_{n+1}}{a_n} < \frac{b_{n+1}}{b_n}$, AND $\sum b_n$ CONVERGES, $\sum a_n$ CONVERGES

E) CONVEGING & DIVEGING SERIES FOR COMPARISON

1) $\sum_{n=1}^{\infty} \frac{1}{n^p} (p \leq 1)$ DIVERGES (HARMONIC SERIES $p=1$)

2) $\sum_{n=1}^{\infty} \frac{1}{n^p}$, $(p > 1)$ CONVERGES 4) $\frac{1}{n!}$ CONVERGES

3) ALL TRIG. FUNCTIONS DIVERGE

F) CONVERGES IF $\frac{a_{n+1}}{a_n} < r < 1$, AND DIVERGES IF $\frac{a_{n+1}}{a_n} > r > 1$

G) GEOMETRIC SERIES

1) $\lim_{n\to\infty} S_n = \frac{a}{1-r}$

2) $\lim S_n = \frac{a(1-r^n)}{1-r}$

x 1) $\int_0^{\pi}\int_0^{x} x \sin Y \, dY \, dx$

$\int_0^{\pi} -x \cos Y \Big]_0^{x} dx$

$\int_0^{\pi} [-x(\cos x - 1)] dx$

$\int_0^{\pi} (-x \cos x + x) dx$

$Q = -\int x \cos x$

$dV = \cos x \quad U = x$

$V = -\sin x \quad dU = dx$

$Q = -x \sin x + \int \sin x \, dx$

$= -x \sin x - \cos x$

$\left[x \sin x - \cos x + \frac{x^2}{2} \right]_0^{\pi}$

$\left(1 + \frac{\pi^2}{2}\right) - 1$

$\frac{\pi^2}{2}$

x2) $\int_1^{\ln 8} \int_0^{\ln Y} e^{x+Y} \, dx \, dY$

$\int_1^{\ln 8} \left[e^{x+Y} \right]_0^{\ln Y} dY$

$\int_1^{\ln 8} e^{\ln Y} e^{Y} dY$

$\int_1^{\ln 8} Y e^{Y} dY$

$dV = e^{Y} \quad U = Y$

$V = e^{Y} \quad dU = dY$

$Ye^{Y} - \int e^{Y} dY$

$\left[Ye^{Y} - e^{Y} \right]_1^{\ln 8}$

$(8 \ln 8 - 8) - (e - e)$

$8 \ln 8 - 8$

3) $\int_0^{\pi} \int_0^{\sin x} Y \, dY \, dx$

$\int_0^{\pi} \frac{Y^2}{2} \Big]_0^{\sin x} dx$

$\int_0^{\pi} \frac{\sin^2 x}{2} dx$

$\int_0^{\pi} \frac{1}{2} \left[\frac{1}{2} (1 - \cos 2x) \right] dx$

$\frac{1}{4} \int_0^{\pi} (1 - \cos 2x) dx$

$\frac{1}{4} \left[x - \frac{1}{2} \sin 2x \right]_0^{\pi}$

$\frac{1}{4} (\pi)$

4) $\int_1^{2} \int_Y^{Y^2} dx \, dY$

$\int_1^{2} [x]_Y^{Y^2} dY$

$\int_1^{2} (Y^2 - Y) dY$

$\left[\frac{Y^3}{3} - \frac{Y^2}{2} \right]_1^{2}$

$\left(\frac{8}{3} - 2 \right) - \left(\frac{1}{3} - \frac{1}{2} \right)$

$\frac{16 - 12 - 2 + 3}{6}$

$\frac{5}{6}$

$\ln Y) dY$

$(e^{2x}-2)-(1)$ $\quad$ $[2Y-\frac{1}{Y}]_1^{e^2}$

$e^{2x}-3$ $\quad$ $(2e^2-e^{-2})-(2-1)$

6) $\int_0^1 \int_{\sqrt{Y}}^1 dx\,dY = \int_0^1 \int_0^{x^2} dY\,dx$

7) $\int_0^{\sqrt{2}} \int_{-\sqrt{4-2Y^2}}^{\sqrt{4-2Y^2}}$

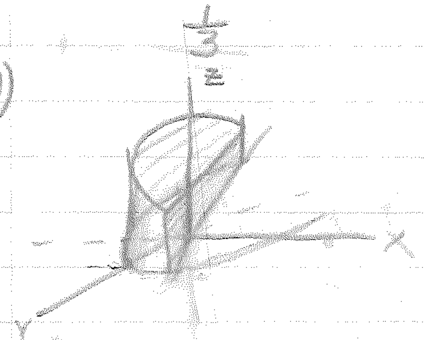
6) $\int_0^1 \int_{\sqrt{Y}}^1 dx\,dY$

$\int_0^1 (1-\sqrt{Y})\,dY$

$[Y-\frac{2}{3}Y^{\frac{3}{2}}]_0^1$

$\frac{1}{3}$

9)



$Y = 4-x^2$ $\quad$ $z = x+4$

$Y = 3x$ $\quad$ $x = \sqrt{4-Y}$

$V = \int\int_0^{\sqrt{4-Y}} (x+4)\,dx\,dY$

$V = \int_3^{-12}\int_0^{\sqrt{4-Y}} (x+Y)\,dx\,dY$

$4-x^2 = 3x$

$-x^2 = 3x-4$

$x^2+3x-4=0$

$x = \frac{-3\pm 5}{2}$ $\quad$ $Y=3X$

$x=1$ $\quad$ $x=-4$

$Y=3$ $\quad$ $Y=-12$

$\int_3^{-12} \frac{x^2}{2} + Yx]_0^{\sqrt{4-Y}}\,dY$

$\int_3^{-12} \left(\frac{4-Y}{2} + Y\sqrt{4-Y}\right) dY$

$\frac{3Y}{2}$

$\frac{1}{2} - \frac{Y}{2} + 2Y - Y^{\frac{3}{2}}$

$\left(\frac{1}{2}Y - \frac{3Y^2}{4} - \frac{2}{5}Y^{\frac{5}{2}}\right]_3^{-12}$

x 1) $\int_0^{\pi}\int_0^{x} x \sin Y \, dY \, dx$

$\int_0^{\pi} -x\cos Y\Big]_0^{x} dx$

$\int_0^{\pi} \left[x(\cos x - 1)\right] dx$

$\int_0^{\pi} (-x\cos x + x)\, dx$

$Q = \int x \cos x$

$dV = \cos x \quad U = x$

$V = -\sin x \quad dU = dx$

$Q = -x\sin x + \int \sin x \, dx$

$= -x \sin x - \cos x$

$\left[x \sin x - \cos x + \frac{x^2}{2}\right]_0^{\pi}$

$\left(1 + \frac{\pi^2}{2}\right) - 1$

$\frac{\pi^2}{2}$

x2) $\int_1^{\ln 8}\int_0^{\ln Y} e^{x+Y} \, dx \, dY$

$\int_1^{\ln 8} \left[e^{x+Y}\right]_0^{\ln Y} dY$

$\int_1^{\ln 8} e^{\ln Y} e^{Y} \, dY$

$\int_1^{\ln 8} Y e^{Y} dY$

$dV = e^{Y} \quad U = Y$

$V = e^{Y} \quad dU = dY$

$Ye^{Y} - \int e^{Y} dY$

$\left[Ye^{Y} - e^{Y}\right]_1^{\ln 8}$

$(8\ln 8 - 8) - (e - e)$

$8 \ln 8 - 8$

3) $\int_0^{\pi}\int_0^{\sin x} Y \, dY \, dx$

$\int_0^{\pi} \frac{Y^2}{2}\Big]_0^{\sin x} dx$

$\int_0^{\pi} \frac{\sin^2 x}{2} \, dx$

$\int_0^{\pi} \frac{1}{2}\left[\frac{1}{2}(1 - \cos 2x)\right] dx$

$\frac{1}{4}\int_0^{\pi} (1 - \cos 2x)\, dx$

$\frac{1}{4}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi}$

$\frac{1}{4}(\pi)$

4) $\int_1^{2}\int_Y^{Y^2} dx \, dY$

$\int_1^{2} [x]_Y^{Y^2} dY$

$\int_1^{2} (Y^2 - Y)\, dY$

$\left[\frac{Y^3}{3} - \frac{Y^2}{2}\right]_1^{2}$

$\left(\frac{8}{3} - 2\right) - \left(\frac{1}{3} - \frac{1}{2}\right)$

$\frac{16 - 12 - 2 + 3}{6}$

$\frac{5}{6}$

5) $\int_0^2 \int_1^{e^x} dY\, dx = \int_1^{e^2} \int_{\ln Y}^{2} dx\, dY$

$\int_0^2 (e^x - 1)\, dx = \int_1^{e^2} (2 - \ln Y)\, dY$

$\left[e^x - x\right]_0^2 \qquad \left[2Y - \frac{1}{Y}\right]_1^{e^2}$

$(e^{2x} - 2) - (1) \qquad (2e^2 - e^{-2}) - (2 - 1)$

$e^{2x} - 3$

6) $\int_0^1 \int_{\sqrt{Y}}^{1} dx\, dY = \int_0^1 \int_0^{x^2} dY\, dx$

7) $\int_0^{\sqrt{2}} \int_{-\sqrt{4-2Y^2}}^{\sqrt{4-2Y^2}}$

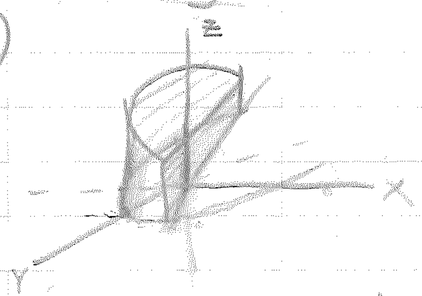
6) $\int_0^1 \int_{\sqrt{Y}}^{1} dx\, dY$

$\int_0^1 (1 - \sqrt{Y})\, dY$

$\left[Y - \frac{2}{3} Y^{\frac{3}{2}}\right]_0^1$

$\frac{1}{3}$

9)



$Y = 4 - x^2 \qquad z = x + 4$

$Y = 3x \qquad x = \sqrt{4 - Y}$

$V = \int \int_0^{\sqrt{4-Y}} (x + 4)\, dx\, dY$

$V = \int_3^{-12} \int_0^{\sqrt{4-Y}} (x + Y)\, dx\, dY$

$4 - x^2 = 3x$

$-x^2 = 3x - 4$

$x^2 + 3x - 4 = 0$

$x = \frac{-3 \pm 5}{2} \qquad Y = 3X$

$X = 1 \quad X = -4$

$Y = 3 \quad Y = -12$

$\int_3^{-12} \left.\frac{x^2}{2} + Yx\right]_0^{\sqrt{4-Y}} dY$

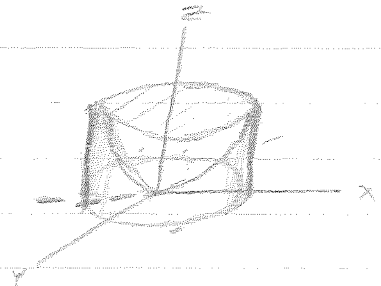
$\int_3^{-12} \left(\frac{4-Y}{2} + Y\sqrt{4-Y}\right) dY$

$\frac{3Y}{2}$

$\frac{1}{2} - \frac{Y}{2} + 2Y - Y^{\frac{3}{2}}$

$\left(\frac{1}{2}Y - \frac{3Y^2}{4} - \frac{2}{5}Y^{\frac{5}{2}}\right]_3^{-12}$

19)



$x^2+y^2=a^2 \qquad az = x^2+y^2$

$$\frac{1}{2}V = \int_0^a\int_0^{\sqrt{a^2-x^2}}\left(\frac{x^2}{a}+\frac{y^2}{a}\right)dy\,dx$$

$$= \int_0^a \left.\frac{x^2y}{a} + \frac{y^3}{3a}\right]^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \left(\frac{x^2\sqrt{a^2-x^2}}{a} + \frac{(a^2-x^2)^{\frac{3}{2}}}{3a}\right)dx$$

$\frac{1}{a}\int x^2(a^2-x^2)^{\frac{1}{2}}dx$

$u = x^2 \qquad dv = (a^2-x^2)^{\frac{1}{2}}$

$du = 2x \qquad p = a^2-x^2 \qquad v = \int p^{\frac{1}{2}}dx$

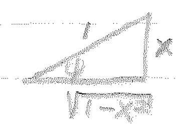
$x = (a^2-p)^{\frac{1}{2}} \qquad v = \int\left(\frac{p}{a^2-p}\right)dp$

$dx = \frac{-1}{\sqrt{a^2-p}}$

$\frac{1}{a}\int_0^a x^2(a^2-x^2)^{\frac{1}{2}}dx \qquad x = \cos\varphi$

$dx = -\sin\varphi\, d\varphi$

$-\frac{1}{a}\int_{\frac{\pi}{2}}^{\cos^{-1}a} \cos^2\varphi \sin^2\varphi\, d\varphi$



$-\left.\frac{\sin\varphi\cos^3\varphi}{4}\right]_{\frac{\pi}{2}}^{\cos^{-1}a}$

12) $\int_0^{\pi}\int_x^{\pi} \frac{\sin Y}{Y}\, dY\, dx$

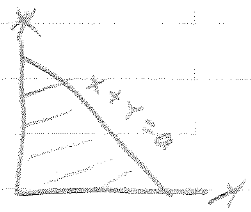
$U = \sin Y \quad dv = \frac{1}{Y}$ $\qquad$ $\tan Y$

$du = \cos Y \quad V = \ln Y$

$\sin Y \ln Y - \int \cos Y \ln Y$

16.3

✓ 1)



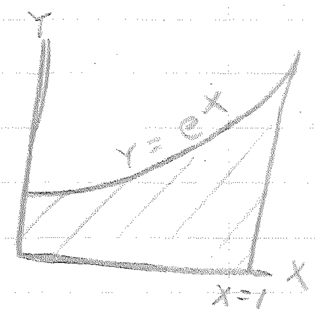
$\int_0^a \int_0^{a-Y} dx\, dY$

$\int_0^a (a-Y)\, dY$

$aY - \frac{Y^2}{2} \Big]_0^a$

$a^2 - \frac{1}{2}a^2 = \frac{1}{2}a^2$

2)



$\int_1^e \int_0^{\ln Y} dx\, dY$

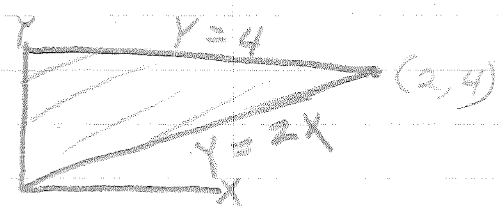
$\int_0^1 \int_1^{e^x} dY\, dx$

$\int_0^1 (e^x - 1)\, dx$

$(e^x - x\Big]_0^1$

$(e-1)-1 = e-2$

✓ 3)

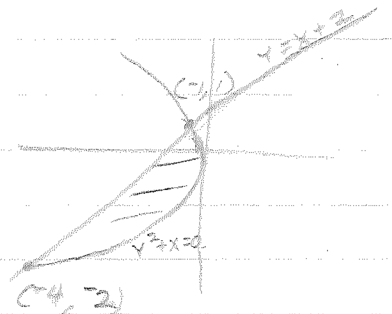


$\int_0^4 \int_0^{\frac{Y}{2}} dx\, dY$

$\int_0^4 \frac{Y}{2}\, dY$

$\frac{Y^2}{4}\Big]_0^4 = 4$

4)



$$\int_{-4}^{0}\int_{x+2}^{-\sqrt{x}} dy\ dx$$

$$\int_{-2}^{1}\int_{y-2}^{-y^2} dx\ dy$$

$$\int_{-2}^{1} (-y^2 - y + 2)\, dy$$

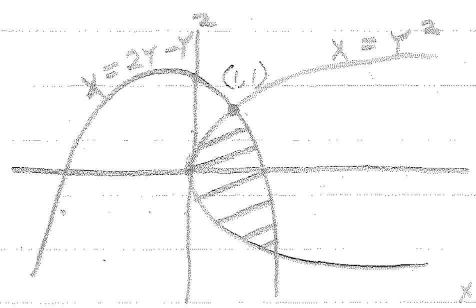
$$\frac{-y^3}{3} - \frac{y^2}{2} + 2y \Big]_{-2}^{1}$$

$$\left[\frac{-1}{3} - \frac{1}{2} + 2\right] - \left[\frac{8}{3} - \frac{4}{2} - 4\right]$$

$$\frac{-2-3+12}{6} \qquad \frac{16-12-24}{6}$$

$$1 + \frac{20}{6} = \frac{26}{6} = \frac{13}{3}$$

5)



$2y - y^2 = y^2$ (0, 0)

$2y = 2y^2$ (1, 1)

$y = 1, 0$

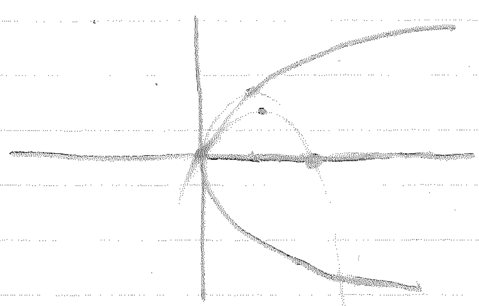
$2y - y^2 = y^2$

$x = 2y - y^2$

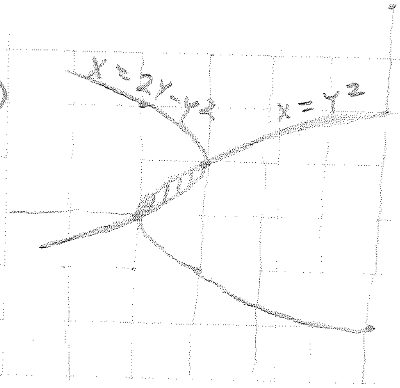
$2y = 2y^2$

$y = y^2$

$y = 0, 1$



✓5)



$x = y^2$

$x = 2y - y^2$

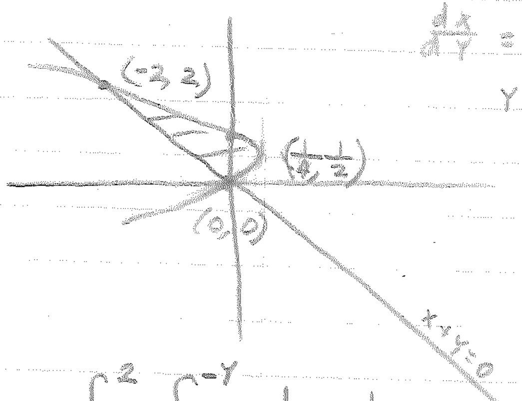
$$\int_0^1 \int_{2y-y^2}^{y^2} dx\, dy$$

$$\int_0^1 (y^2 - 2y + y^2)\, dy$$

$$2\left[\frac{y^3}{3} - \frac{y^2}{2}\right]_0^1$$

$$2\left[\frac{1}{3} - \frac{1}{2}\right] = \frac{1}{3}$$

✓7)



$\frac{dx}{dy} = 1 - 2y = 0 \Rightarrow y = \frac{1}{2}$

$y - y^2 = -y$

$y^2 = 2y$

$y = 2$

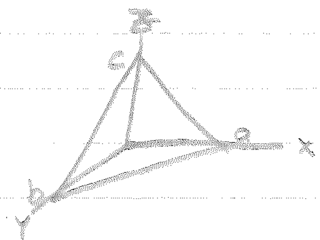
$$\int_0^2 \int_{y-y^2}^{-y} dx\, dy$$

$$\int_0^2 (-y - y + y^2)\, dy$$

$$-y^2 + \frac{y^3}{3}\Big]_0^2$$

$$-4 + \frac{8}{3} = \frac{4}{3}$$

x 1) $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ $\quad \int_0^c \int_0^{b(1-\frac{z}{a})} \int_0^{a(1-\frac{y}{b}-\frac{z}{c})} dx\, dy\, dz$



$$a \int_0^c \int_0^{b(1-\frac{z}{a})} \left(1 - \frac{y}{b} - \frac{z}{c}\right) dy\, dz$$

$$a \int_0^c \left[y - \frac{y^2}{2b} - \frac{zy}{c} \right]_0^{b(1-\frac{z}{a})} dz$$

$$a \int_0^c \left[b\left(1-\frac{z}{a}\right) - \frac{b^2\left(1-\frac{z}{a}\right)^2}{2b} - \frac{bz\left(1-\frac{z}{a}\right)}{c} \right] dz$$

$$a \int_0^c \left[b - \frac{b}{a} z - \frac{b\left(1 - 2\frac{z}{a} + \frac{z^2}{a^2}\right)}{2} - \frac{b}{c} z + \frac{bz^2}{ac} \right] dz$$

$$a \int_0^c \left[b - \frac{b}{a} z - \frac{b}{2} + \frac{bz}{a} - \frac{bz^2}{2a^2} - \frac{b}{c} z + \frac{bz^2}{ac} \right] dz$$

$$a \left[bz - \frac{b}{2a} z^2 - \frac{b}{2} z + \frac{bz^2}{2a} - \frac{bz^3}{3a^2} - \frac{b}{2c} z^2 + \frac{bz^3}{3ac} \right]_0^c$$

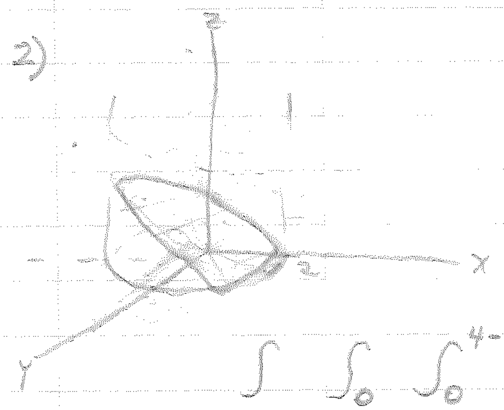
$$a \left[z^3\left(\frac{-b}{3a^2} + \frac{b}{3ac}\right) + z^2\left(\frac{-b}{2a} + \frac{b}{2a} - \frac{b}{2c}\right) + z\left(b - \frac{b}{2}\right) \right]_0^c$$

$$a \left[\cancel{c^3\left(\frac{-b}{3ac} + \frac{b}{3ac}\right)} + c^2\left(\frac{-b}{2c}\right) + c\frac{b}{2} \right]$$

$$a - cb/2 + \frac{cb}{2}$$

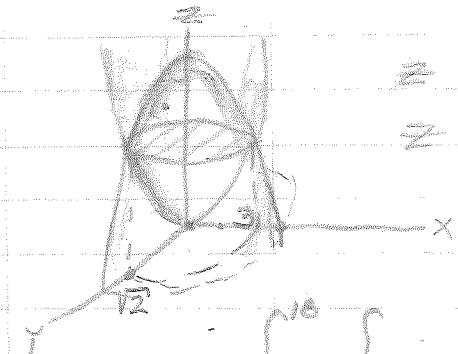
$$a$$

X 2)



$$\int \int_0 \int_0^{4-y^2} dx\, dy\, dz$$

3)



$z = x^2 + 9y^2$

$z = 18 - x^2 - 9y^2$

$$\int_0^{18} \int \int_0^{\sqrt{z-18+9y^2}} dx\, dy\, dz$$

$z - 9y^2 = 18 - z - 9y^2$

$2z = 18$

$z = 9$

$x^2 + 9y^2 = 18 - x^2 - 9y^2$

$2x^2 + 9y^2 = 18$

$y = \frac{18-2x^2}{9}$

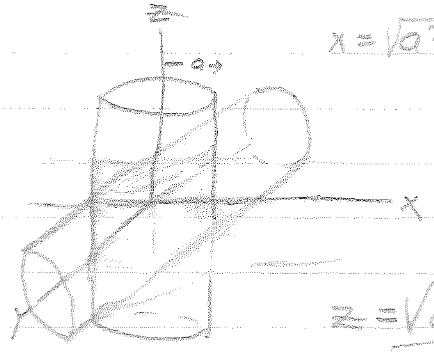
$$\int_0^3 \int_0^{(18-2x^2)/9} \int_0^{18-x^2-9y^2} dz\, dy\, dx$$

$$\int_{-3}^3 \int_0^{\frac{18-2x^2}{9}} 18 - x^2 - 9y^2\, dy\, dz$$

$$(18-x^2)y - 3y^3\Big]$$

$$\int_{-3}^3 (18-x^2)\frac{18-2x^2}{9} - 3\left(\frac{18-2x}{9}\right)^2$$

4)



$x = \sqrt{a^2 - y^2}$

$$\int_0^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} dz\,dy\,dx$$

$z = \sqrt{a^2 - x^2}$

$z = \sqrt{a^2 - x^2}$

$$2 \int_0^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}\;\; z = a} (a^2 - y^2)^{\frac{1}{2}}\, dy\, dx$$

15-14

1) $\int_x^{x^2} t^{-1}\,dt$

$\ln \cancel{t}\Big]_x^{x^2}$

$2\ln x^2 - \ln x$

$\frac{2}{x} - \frac{1}{x}$

$\frac{1}{x}$

7) $Y=(e^x)^x$

$\frac{dY}{dX} = 2X\, e^{x^2}$

8) $Y=(X^2+1)^{\frac{1}{x}}$

$U=x^2+1 \quad Y=U^{\frac{1}{x}}$

$2X(X^2+1)^{\frac{1}{x}-1}$

15-13

$(2X^4 + 2XY^3)\,dX \qquad 3Y^2X^2 + 3Y^4$

$\frac{dN}{dX} = 8X^3 + 2Y^3 \qquad \frac{dM}{dY} = 6YX^2 + 12Y^3$

$\frac{dN}{dY} = 2XY^2 \qquad \frac{dM}{dX} = 6Y^2X$

16.2 16.3

1) 

$\sigma \frac{1}{2} a^2$

$\sigma \int_0^a \int_0^{a-y} y \, dx \, dy$

$\sigma \int_0^a (ya - y^2) \, dy$

$\sigma \left[\frac{y^2 a}{2} - \frac{y^3}{3} \right]_0^a$

$\sigma \left(\frac{a^3}{2} - \frac{a^3}{3} \right)$

$M_y = \left(\frac{a}{6} \sigma \right)$

$\sigma \int_0^a \int_0^{a-x} x \, dy \, dx$

$M_x = \frac{a}{6} \sigma$

$M_x = \int_0^a \int_0^{a-x} y \, dy \, dx$

$\int_0^a \frac{(a-x)^2}{2} \, dx$

$u = a - x$

$du = -dx$

$M_x = \int_0^a \frac{u^2}{2} \, dx$

$-\frac{u^3}{6}$

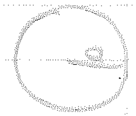
$\frac{(x-a)^3}{6}$

16.5

16-4

1) $\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\,dx$

$2\int_{-a}^{a} \sqrt{a^2-x^2}\,dx$



$r=a$

$\int_0^{2\pi}\int_0^{a} r\,dr\,d\theta$

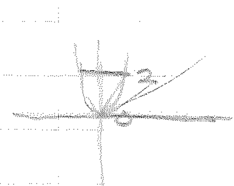
2) $\int_0^{a} \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dx\,dy$

$\frac{x^3}{3}+y^2x$

$\int_0^{a} \frac{(a^2-y^2)^{\frac{3}{2}}}{3}+y^2$

$\int_0^{2\pi}\int_0^{a} r\,dr\,d\theta$

5) $\int_0^{2}\int_0^{x} y\,dy\,dx$



$\int_2^{0} \frac{x^2}{2}\,dx$

$y = \frac{x^2}{2}$

$r\sin\theta = \frac{r^2\cos^2\theta}{2}$

$r = \frac{\tan\theta\cos\theta}{2}$

$\int_0 \int_0 r\,dr\,d\theta$

1) $\int_0^{\pi} \int_0^{x} x \sin Y \, dY \, dX$

$\int_0^{\pi} -x \cos Y \Big]_0^{x} \, dx$

$\int_0^{\pi} (-x \cos x + x) \, dx$

$\left[\frac{x^2}{2}\right]_0^{\pi} - \int x \cos x \, dx$

$\frac{\pi^2}{2} - \int_0^{\pi} x \cos x \, dx$

7) $\int_0^{\sqrt{2}} \int_{-\sqrt{4-2Y^2}}^{\sqrt{4-2Y^2}} Y \, dx \, dY$

$\int_0^{\sqrt{2}} (4-2Y^2) \, dY$

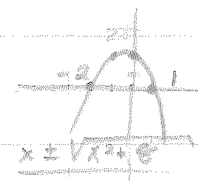
$Y = \sqrt{\frac{4-x}{2}}$

$\int_0^{4} \int_0^{\sqrt{(4-x)/2}} dY \, dx$

8) $\int_{-2}^{1} \int_{x^2+4x}^{3x+2} dY \, dx$

$\int_{-2}^{1} (3x+2-x^2-4x) \, dx$

$\int_{-2}^{1} (-x^2 - x + 2) \, dx$



$\int_0^{\frac{9}{4}} \int dx \, dY$

$Y = \frac{x \pm \sqrt{x^2+8}}{-2}$

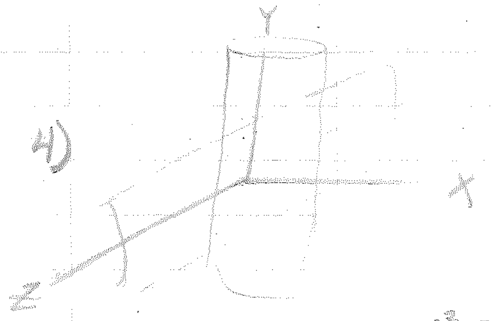
$\frac{1}{4} + \frac{1}{2} + 2 \quad 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$

16.4

1) $\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}$

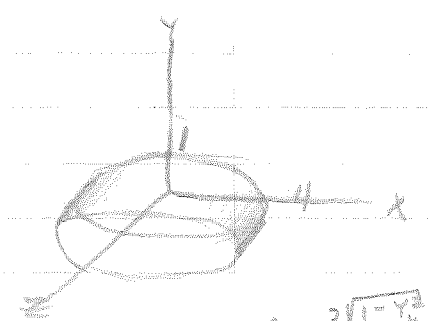
$\sqrt{a^2 - x^2} = \sqrt{a^2 - r^2\cos^2\theta}$

4)



$$\int \int \int^{a^2-z} dx\, dy\, dz$$

6)



$$\int_{-1}^{1} \int_{-2\sqrt{1-y^2}}^{2\sqrt{1-y^2}} \int_{0}^{x+2} dz\, dx\, dy$$

$$x = \sqrt{4-4y^2} = 2\sqrt{1-y^2}$$

$$\int_{-1}^{1} \int_{-2\sqrt{1-y^2}}^{2\sqrt{1-y^2}} (x+2)\, dx\, dy$$

$$\left.\frac{x^2}{2} + 2x\right]_{-2\sqrt{1-y^2}}^{2\sqrt{1-y^2}}$$

Pg 545

66) $\sin(X+Y) + \sin(Y+Z) = 1$

$\sin(Y+Z) = 1 - \sin(Y+X)$

$Y + Z = \sin^{-1}(1 - \sin(Y+X))$

$Z = -Y + \sin^{-1}(1 - \sin(X+Y))$

$V = \sin U$

$\frac{dV}{dU} = \cos U$

$\frac{dU}{dV} = \frac{d}{dV}(\sin^{-1} V) = \sec U = \frac{1}{\sqrt{1-V^2}}$

$Q = (1 - \sin(X+Y)) \Rightarrow \frac{\partial Q}{\partial X} = -\cos(X+Y)$

$Z = -Y + \sin^{-1} Q$

$\frac{\partial Z}{\partial Q} = -1 + \frac{1}{\sqrt{1-Q}}$

$\frac{\partial Z}{\partial X} = -\cos(X+Y)\left[-1 + (\sin(X+Y))^{-\frac{1}{2}}\right]$

$\sin^{-1}(1-Q) - X = Y$

$Z = X - \sin^{-1}(1-Q) + \sin^{-1} Q$

$\frac{\partial Z}{\partial Q} = 1 + \frac{1}{\sqrt{Q}} + \frac{1}{\sqrt{1-Q}}$

$\frac{\partial Z}{\partial Q} = 1 + (1 - \sin(X+Y))^{-\frac{1}{2}} + (\sin(X+Y))^{-\frac{1}{2}}$

$\frac{\partial Z}{\partial X} = \left[1 + (1 - \sin(X+Y))^{-\frac{1}{2}} + (\sin(X+Y))^{-\frac{1}{2}}\right] - \cos(X+Y)$

$\frac{\partial Z}{\partial X}\,\frac{\partial Z}{\partial Y} = \frac{\partial^2 Z}{\partial Y \partial X}$

$\frac{\partial^2 Z}{\partial Y \partial X} = \cos(X+Y)\left[1 + (1 - \sin(X+Y))^{-\frac{1}{2}} + (\sin(X+Y))^{-\frac{1}{2}}\right]$
$\cdot \left[(\sin(X+Y))^{-\frac{1}{2}} - 1\right]$

14-4) c) $x + Y = \int_0^x f(x)\,dx$

$1 + Y' = Y$

$Y' - Y = 1$

$Y = e^{+x}\int e^{-x}dx + ce^{x}$

$= -e^{+x}e^{-x} + ce^{x}$

$= ce^{x} + 1$

$2 = c + 1$

$c = 1$

$Y = e^{x} + 1$

~~14-5) a)~~ $Y' - 2xY = x^3$

$(D - 2x)Y = x^3$

$\cancel{Y = e^{2x^2}\int e^{-2x^2}}$

$\cancel{Y = e^{2x^2}}$

$e^{2x^2}D(e^{-2x^2}Y) = x^3$

$D(e^{-2x^2}Y) = x^3 e^{-2x^2}$

$e^{-2x^2}Y = \int x^3 e^{-2x^2}dx + e^{2x^2}$

$-\frac{1}{2}x^3e^{-2x} + \int 2x^2 e^{-2x}dx$

$+ x^2(-\frac{1}{2})e^{-2x} + \int x e^{-2x}$

$+ x(-\frac{1}{2}e^{-2x}) + \int \frac{e^{-2x}}{2}$

$\cancel{e^{-2x^2}}\,Y = -\frac{1}{2}x^3\cancel{e^{-2x}} - \frac{1}{2}x^2\cancel{e^{-2x}} - \frac{1}{2}x\cancel{e^{-2x}} + ce^{2x} - \frac{e^{-2x}}{4}$

$= -\frac{1}{2}\left[x^3 - x^2 - x \mp \frac{1}{2}\right] + ce^{2x}$

14-5) a) $Y' - 2xY = x^3$

$D(Y - 2x) = x^3$

$e^{2x^2} D(e^{-2x^2} Y) = x^3$

$D(e^{-2x^2} Y) = x^3 e^{2x^2}$

$e^{+2x^2} Y = \int x^3 e^{-2x^2} dx + Ce^{+2x^2}$

$= x^3 \frac{e^{-2x^2}}{-4x} + \int 3x^2 \frac{e^{-2x^2}}{4x}$

$= -\frac{x^2 e^{-2x^2}}{4} + \int \frac{3x e^{-2x^2}}{4}$

$\cancel{+ 3x\left(\frac{e^{-2x^2}}{-4x}\right) - \int \frac{3e^{-2x^2}}{4}}$

$\cancel{= \frac{3e^{-2x^2}}{4} - \frac{3e^{-2x^2}}{16}}$

$-4x \quad \frac{3x}{16} e^{-2x^2}$

$Y = \frac{-x^2}{4} + \frac{3x}{16} + Ce^{2x^2}$

6

BOB MARKS
CALC III

7-1) $Y' + 2Y = 0$

$$Y = e^{-ax} \int e^{2x} f(x)\,dx + e^{-ax}$$

$$= 0 + Ce^{-2x}$$

7-2) $Y' + 2Y = 3$

$$Y = e^{-2x} \int e^{2x}\, 3\,dx + Ce^{-2x}$$

$$= e^{-2x}\, \frac{3}{2} e^{2x} + Ce^{-2x}$$

$$= Ce^{-2x} + \frac{3}{2}$$

7-3) $Y' + 2Y = ax + b$

$$Y = e^{-2x} \int e^{2x}(ax+b) + Ce^{-2x}$$

$$\left(\frac{1}{2}e^{2x}\right)(ax+b) - \frac{a}{4}e^{2x} + Ce^{-2x}$$

7-4) $Y' + aY = e^{bx}$

$$Y = e^{-ax} \int e^{bx} e^{ax}\,dx + Ce^{-ax}$$

$$= e^{-ax} \int e^{x(b+a)}\,dx + Ce^{-ax}$$

$$= e^{-ax}\, \frac{e^{x(b+a)}}{b+a} + Ce^{-ax}$$

$$= \frac{e^{bx}}{b+a} + Ce^{-ax}$$

7-5) $Y' - 2Y = \sin X$

$$Y = e^{2x} \int e^{-2x} \sin X + Ce^{2x}$$

$$= e^{2x}\left(\frac{e^{-2x}(-\cos X - 2\sin X)}{5}\right) + Ce^{2x}$$

$$= \frac{-\cos X - 2\sin X}{5} + Ce^{2x}$$

$$1 = \frac{-1}{5} + C$$

$$C = \frac{6}{5}$$

$$Y = \frac{-\cos X - 2\sin X}{5} + \frac{6}{5}e^{2x}$$

~~12-2)~~ $y' - y = x^2 + 5$

$y = e^{x} \int e^{-x}(x^2+5)\,dx + ce^{x}$

$= e^{x}\left[(-e^{-x}(2x)) + \int (2x)e^{-x}\,dx\right] + ce^{x}$

$= e^{x}\left[2xe^{-x} - (2x)e^{-x} + \int 2e^{-x}\,dx\right] + ce^{x}$

$= e^{x}\left[-2xe^{-x} - 2xe^{-x} + 2e^{-x}\right] + ce^{x}$

$= -2x - 2x - 2 + ce^{x}$

$= -4x - 2 + ce^{x}$

12-2) $y' - y = x^2 + 5$

$-2x - 2 + x^2 + 2x + 7 = x^2 + 5$

13-1) a) $y' - 4y = 0$

$y = e^{4x} \int e^{-4x}(0)\,dx + ce^{4x}$

$= ce^{4x}$

b) $2y' + 5y = 2$

$y' + \frac{5}{2}y = 1$

$y = e^{\frac{5}{2}} \int e^{\frac{5}{2}}\,dx + ce^{-\frac{5}{2}}$

$= e^{\frac{5}{2}}(0) + ce^{-\frac{5}{2}}$

$= ce^{-\frac{5}{2}}$

c) $r' + 3r = 2 - \theta$

$r = e^{-3\theta} \int e^{3\theta}(2-\theta)\,d\theta + ce^{-3\theta}$

$= e^{-3\theta}\left[\frac{1}{3}e^{3\theta}(2-\theta) + \int + \frac{1}{3}e^{3\theta}\,d\theta\right] + ce^{-3\theta}$

$= \text{''} \left[\frac{1}{3}e^{3\theta}(2-\theta) + \frac{1}{9}e^{3\theta}\right] + ce^{-3\theta}$

$= \frac{1}{3}(2-\theta) + \frac{1}{9} + ce^{-3\theta}$

$= \frac{2}{3} - \frac{\theta}{3} + \frac{1}{9} + ce^{-3\theta}$

$= \frac{5}{9} - \frac{\theta}{3} + ce^{-3\theta}$

13-1) d) $Y' + 3Y = (2-x)e^{-3x}$

$$Y = e^{-3x}\int e^{3x}(2-x)e^{-3x}\,dx + ce^{-3x}$$

$$\int(2-x)\,dx$$

$$e^{-3x}\left(2x - \frac{x^2}{2}\right) + ce^{-3x}$$

$$= e^{-3x}\left(2x - \frac{x^2}{2} + c\right)$$

e) $s' = 2t - s$

$$s' + s = 2t$$

$$s = e^{-t}\int e^{t}(2t)\,dt + ce^{-t}$$

$$2te^{t} - \int 2e^{t}\,dt$$

$$= e^{-t}(2te^{t} - 2e^{t}) + ce^{-t}$$

$$= 2t - 2 + ce^{-t}$$

f) $Y'' - 2Y' = 2x$

$$Y' - 2Y = x^2$$

$$Y = e^{+2t}\int e^{-2t}t^2\,dt + ce^{2t}$$

$$\left(t^2 - \frac{1}{2}e^{-2t} - \int 2t \cdot -\frac{1}{2}e^{-2t}\right)$$

$$+ \int t e^{-2t}$$

$$-t \cdot \frac{1}{2}e^{-2t} + \int \frac{1}{2}e^{-2t}$$

$$-\frac{1}{4}e^{-2t}$$

$$Y = e^{2t}\left[t^2 \frac{-e^{-2t}}{2} - t\,\frac{e^{-2t}}{2} - \frac{1}{4}e^{-2t}\right] + ce^{2t}$$

$$= -\frac{t^2}{2} - \frac{t}{2} - \frac{1}{4} + ce^{2t}$$

13-2) a) $Y' - 4Y = 0$

$$Y = e^{4x}\int e^{-4x}(0) + ce^{4x}$$

$$Y = ce^{+4x}$$

$$1 = c(1)$$

$$Y = e^{4x}$$

b) $Y' + 3Y = e^{-x}$

$$Y = e^{-3x}\int e^{3x}e^{-x}\,dx + ce^{-3x}$$

$$= e^{-3x}\int e^{2x}\,dx + ce^{-3x}$$

$$= e^{-3x}\,\frac{1}{2}e^{2x} + ce^{-3x}$$

$$= e^{-x}\left(\frac{1}{2}\right) + ce^{-3x}$$

$$0 = \frac{1}{2e} + \frac{c}{e^3}$$

$$-\frac{1}{2e} = \frac{c}{e^3}$$

$$-\frac{e^2}{2} = c$$

$$Y = \frac{e^{-x}}{2} - \frac{e^2}{2}e^{-3x}$$

$$= \frac{e^{-x}}{2} - \frac{e^{2-3x}}{2}$$

c) $Y' + 3Y = 2-x$

$$Y = e^{-3x}\int e^{3x}(2-x)\,dx + ce^{-3x}$$

$$(2-x)\frac{1}{3}e^{3x} + \int \frac{e^{3x}}{3}\,dx$$

$$Y = e^{-3x}\left[\frac{1}{3}(2-x)e^{3x} + \frac{e^{3x}}{9}\right] + ce^{-3x}$$

$$= \frac{1}{3}(2-x) + \frac{1}{9} + ce^{-3x}$$

$$\frac{1}{3} + \cancel{\frac{1}{9}} + ce^{-3} = -1 + \cancel{\frac{1}{9}} + ce^{3}$$

$$\frac{4}{3} = c\left(e^{3} - e^{-3}\right)$$

$$Y = \frac{7}{9} - \frac{x}{3} + \frac{4}{(e^3 - e^{-3})3}e^{-3x}$$

13-2) d) $\frac{dQ}{dt} + \frac{3}{100} Q = 0$

$Q' + \frac{3}{100} Q = 0$

$Q = Ce^{-.03t}$

$C = 10$

$Q = 10e^{-.03t}$

13-3) a) $Y' + 2Y = 0$

$Y = Ce^{-2x}$

$0 = Ce^{-2x_0}$

$C = 0 \Rightarrow Y = 0$

~~14-4~~) a) $Y' = 2X + 2Y$

$Y' - 2Y = 2X$

$Y = e^{2x} \int e^{-2x}(2x)\,dx + Ce^{2x}$

$-\frac{1}{2}e^{-2x}(2x) + \int e^{-2x}$

$-e^{2x}X - \frac{1}{2}e^{-2x}$

$= -X - \frac{1}{2} + Ce^{2x}$

$1 = -1 - \frac{1}{2} + Ce^{2}$

$\frac{5}{2} = e^{2}C$

$C = \frac{5}{2e^{2}}$

$Y = -X - \frac{1}{2} + \frac{5}{2e^{2}}e^{2x}$

$= -X - \frac{1}{2} + \frac{5}{2}e^{x}$

14-4) a) $Y' = 2X + 2Y$

$$Y' - 2Y = 2X$$

$$Y = e^{2x} \int \overset{dv}{e^{-2x}}\, \overset{u}{2x}\, dx + ce^{2x}$$

$$-\tfrac{1}{2} 2x e^{-2x} + \int 2 \tfrac{1}{2} e^{-2x}$$

$$- \tfrac{1}{2} e^{-2x}$$

$$= e^{2x} \left[-x e^{-2x} - \tfrac{1}{2} e^{-2x} \right] + ce^{2x}$$

$$= -x - \tfrac{1}{2} + ce^{2x}$$

work

$$\int \overset{dv}{e^{kx}} \overset{u}{(cx^a + b)}\, dx$$

$$(cx^a + b) \tfrac{1}{k} e^{kx} - \int (acx^{(a-1)}) \tfrac{1}{k} e^{kx} dx$$

14-4) b) $Y' = -x - y$

$$Y' + Y = -X$$

$$Y = e^{-x} \int -x e^x dx + ce^{-x}$$

$$-x e^x + \int e^x$$

$$-e^x x + e^x$$

$$= -x + 1 + ce^{-x}$$

$$2 = 1 + c$$

$$c = 1$$

$$Y = 1 - x + e^{-x}$$

12-2) $Y' - Y = X^2 + 5$

$Y = e^{x} \int e^{-x}(X^2+5)\,dx + ce^{x}$

$-(X^2+5)e^{-x} + \int 2X e^{-x}\,dx$

$-2Xe^{-x} + \int 2e^{-x}\,dx$

$-2Xe^{-x} - 2e^{-x}$

$Y_p = -(X^2+5) - 2X - 2$

$= -(X^2 + 2X + 7)$

13-2) b) $Y' + 3Y = e^{-x}$

$Y_p = \int e^{3x} e^{-x}\,dx$

$= \int e^{2x}\,dx$

$= \frac{1}{2} e^{2x}$

$Y = e^{-3x}\left(\frac{1}{2} e^{2x}\right) + ce^{-3x}$

$= \frac{1}{2} e^{-x} + ce^{-3x}$

$0 = \frac{1}{2e} + \frac{c}{e^3}$

$C = \frac{-e^3}{2e} = -\frac{e^2}{2}$

$Y = \frac{1}{2} e^{-x} - \frac{e^2}{2} e^{-3x}$

$= \frac{1}{2} e^{-x} - \frac{e^{2-3x}}{2}$

c) $Y' + 3Y = 2 - X$

$Y_p = e^{-3x} \int e^{3x}(2-x)\,dx$

$\frac{1}{3}(2-X)e^{3x} + \int \frac{1}{3} e^{3x}\,dx$

$+ \frac{1}{9} e^{3x}$

$Y = \frac{1}{3}\left[(2 - X + \frac{1}{3}\right] + ce^{-3x}$

$\left(\frac{4}{9}\right) + ce^{-3} = -\frac{8}{9} + ce^{3}$

$\frac{12}{9} = c(e^3 - e^{-3}) \Rightarrow c = \frac{4}{(e^3 - e^{-3})3}$

14-5) a) $Y' - 2xY = x^3$

$Y_p = e^{2x^2} \int e^{-2x^2} x^3 \, dx$

$\frac{-x^3}{4x} e^{-2x^2} + \int 3x^2 \left(+ \frac{e^{-2x^2}}{4x} \right) dx$

$\frac{-x^2}{4} e^{-2x} + \int \frac{3x \, e^{-2x^2}}{4} \, dx$

$+ \frac{3}{4} \cancel{x} \left(\frac{-e^{-2x^2}}{4\cancel{x}} \right) + \cancel{\int \frac{3}{4} \frac{e^{-2x^2}}{4x}}$

$+ \frac{3e^{-2x^2}}{16}$

$-\left(\frac{x^2}{4} + \frac{3e^{-2x^2}}{16} \right) + Ce^{2x^2}$

~~c)~~ $Y' + \cot(x) \, Y = e^{-x} \csc x$

$Y_p = e^{-x\cot x} \int e^{x \cot x} e^{-x} \csc x \, dx$

$\int e^{x - x\cot x} \csc x \, dx$

$\csc x \, \frac{e^{x - x\cot x}}{1 - \cot x + x\csc^2 x} + \int \csc x \cot x \, \frac{e^{x - x\cot x}}{1 - \cot x + x\csc^2 x}$

~~g)~~ $\frac{dU}{dV} + \frac{mU}{V} = V$

$U' +$

e) $x^2 U' + 2xU = 2$

$U_p = e^{-2x^2} \int e^{2x^2} \, 2$

$= e^{-2x^2} \, \frac{e^{2x^2}}{2x}$

$= \frac{1}{2x}$

$U = \frac{1}{2x} + Ce^{-2x^2}$

$Y' = \frac{1}{2x} + Ce^{-2x^2} = \frac{1}{2x} + C_2 e^{-x^2}$

$Y = \frac{1}{2} \ln x + \frac{C_2 e^{-x^2}}{-2x} + C_3$

~~14-3)~~ f) $r' + \frac{r}{\theta} = \theta \sin\theta$

$$r_p = e^{\theta} \int \overset{dv}{e^{-\theta}}\, \overset{u}{\theta \sin\theta}\, d\theta$$

$$= e^{\theta} \left[\theta \sin\theta\, e^{-\theta} + \int (\theta \cos\theta + \sin\theta) e^{-\theta}\, d\theta \right.$$

$$+ \int \theta \cos\theta\, e^{-\theta} d\theta + \int \sin\theta\, e^{-\theta} d\theta$$

17-1) $Y'' - (a+b)Y' + abY = 0$

$$Y\left(D^2 - (a+b)D + ab\right) = 0$$

$$(D-a)(D-b)Y = 0$$

$$(D-a)Y = 0$$

$$(D-b)Y = Y$$

$$Y = Ce^{ax} \quad \text{etc.}$$

23-1) a) $Y'' + 6Y' + 5Y = 0$

$$(D^2 + 6D + 5)Y = 0$$

$$(D+5)(D+1)Y = 0$$

$$(D+5)Y = 0$$

$$(D+1)Y = ce^{5x}$$

$$Y' + Y = Ce^{5x}$$

$$Y_p = e^{-x} \int e^{x} C e^{5x} dx$$

$$= Ce^{-x} \int e^{6x} dx$$

$$= Ce^{-x} \frac{e^{6x}}{6}$$

$$= C e^{5x}$$

$$Y = Ce^{5x} + C_1 e^{-x}$$

23-1) d) $4y'' + 4y' + y = 0$

$(4D^2 + 4D + 1)y = 0$

$(2D+1)(2D+1)y = 0$

$(2D+1)^2 y = 0$

$(2D+1)\cancel{y} = 0$

$2y' + y = 0$

$y' + \frac{1}{2}y = 0$

$y_p = 0$

$y = ce^{-\frac{x}{2}}$

$\cancel{e)}$ $y'' + 4y' + 5y = 0$

$(D^2 + 4D + 5)y = 0$

$(D+4)(D+1)y = -y$

$(D+4)\underline{Y} = -y$

$(D+1)y = F(x) = \underline{Y}$

$D(e^{4x}\underline{Y}) = -e^{4x}y$

$D(e^x y) = e^x \underline{Y}$

$e^{4x}y$

23-1) e) $Y'' + 5Y' = 0$

$Y' = Ce^{-5x}$

$Y = C_1 e^{-5x} + C_2$

f) $Y'' + 4Y = 0$

$Y' = Ce^{-4x}$

$Y = C_1 e^{-4x} + C_2$

b) $Y'' - 6Y' + 9Y = 0$

$(D^2 - 6D + 9)Y = 0$

$(D-3)^2 Y = 0$

$(D-3)Y = 0$

$Y' - 3Y = 0$

$Y = Ce^{3x}$

h) $2Y'' + 7Y' + Y = 0$

$(2D^2 + 7D + 1)Y = 0$

$(D+1)(2D+5)Y = 4Y$

$(D+1)Y = \bar{Y}$

$(D + \frac{5}{2})\bar{Y} = \cancel{4}\ 2$

$D(e^x Y) = e^x \bar{Y}$

$D(e^{\frac{5}{2}x}\bar{Y}) = 2e^{\frac{5}{2}x}$

$e^{\frac{5}{2}x}\bar{Y} = \frac{4}{5}e^{\frac{5}{2}x} + C$

$(D+1)Y = \frac{4}{5}e^{\frac{5}{2}x} + C$

$Y' + Y = \frac{4}{5}e^{\frac{5}{2}x} + C$ etc

23-1) $\underset{8}{g)}$ $Y'' + 4Y' + 5Y = 0$

$$(D^2 + 4D + 5)Y = 0$$

$$(D+4)(D)Y = -Y$$

$$(D+4)Y = \Upsilon$$

$$(D)\Upsilon = -1$$

$$\Upsilon = -x + C$$

$$Y' + 4Y = -x + C$$

$$Y_p = e^{-4x}\int \overset{dv}{e^{4x}}(\overset{u}{C} - x)\,dx$$

$$(C_1 - x)\frac{e^{4x}}{4} + \int \frac{e^{4x}}{4}$$

$$+ \frac{e^{4x}}{16}$$

$$Y = \frac{C_1 - x}{4} + \frac{1}{16} + Ce^{-4x}$$

j) $Y'' - 4Y' + 7Y = 0$

$$(D^2 - 4D + 7)Y = 0$$

$$(D+7)DY = -7\Upsilon$$

$$(D+7)Y = \Upsilon$$

$$D\Upsilon = -7 \qquad \Upsilon = -7x + C$$

$$Y' + 7Y = -7x + C$$

$$Y_p = e^{-7x}\int \overset{dv}{e^{7x}}(-\overset{u}{7x} + C)$$

$$\frac{e^{7x}}{7}(-7x + C) + \int 7\frac{e^{7x}}{7}$$

$$+ \frac{e^{7x}}{7}$$

$$Y = \frac{1}{7}(-7x + C) + \frac{e^{7x}}{7} + C_2 e^{-7x}$$

$$= -x + C_3 + \frac{e^{7x}}{7} + C_2 e^{-7x}$$

23-2) b) $Y'' + 3Y' + 2Y = x$

$(D^2 + 3D + 2)Y = x$

$(D+2)(D+1)Y = x$

$(D+2)\bar{Y} = x$

$(D+1)Y = \bar{Y}$

$\bar{Y}' + 2\bar{Y} = x$

$\bar{Y} = e^{-2x}\int e^{2x} x\,dx + Ce^{-2x}$

$= x\frac{e^{2x}}{2} - \int \frac{e^{2x}}{2}$

$- \frac{e^{2x}}{4}$

$\bar{Y} = \frac{x}{2} - \frac{1}{4} + Ce^{-2x}$

$Y' - Y = \frac{x}{2} - \frac{1}{4} + Ce^{-2x}$

$Y_p = e^{x}\int e^{-x}\left(\frac{x}{2} - \frac{1}{4} + Ce^{-2x}\right)dx$

$= e^{x}\left[\int \frac{x}{2}e^{-x}dx - \int \frac{e^{-x}}{4}dx + \int Ce^{-3x}dx\right]$

$= e^{x}\left[+\frac{e^{-x}}{4} - C_2e^{-3x} - \frac{x}{2}e^{-x} + \int \frac{1}{2}e^{-x}dx\right]$

$= e^{x}\left[\frac{e^{-x}}{4} - C_2e^{-3x} - \frac{x}{2}e^{-x} - \frac{1}{2}e^{-x}\right]$

$= \frac{1}{4} - C_2e^{-2x} - \frac{(x+1)}{2}$

a) $Y'' + 3Y' + 2Y = 1$

$(D^2 + 3D + 2)Y = 1$

$(D+2)(D+1)Y = 1$

$(D+2)\bar{Y} = 1$

$(D+1)Y = \bar{Y}$

$\bar{Y}' + 2\bar{Y} = 1$

$\bar{Y} = e^{-2x}\int e^{2x}dx + C_1e^{-2x}$

$= \frac{1}{2} + Ce^{-2x}$

$Y' + Y = \frac{1}{2} + Ce^{-2x}$

$Y_p = e^{-x}\int e^{x}\left(\frac{1}{2} + Ce^{-2x}\right)dx$

$= e^{-x}\left[\int \frac{1}{2}e^{x}dx + \int Ce^{-x}dx\right]$

$= e^{-x}\left(\frac{1}{2}e^{x} - Ce^{-x}\right)$

$= \frac{1}{2} - Ce^{-2x}$

23-2)c) $Y'' + 3Y' + 2Y = \cos x$

$(D^2 + 3D + 2)Y = \cos x$

$(D+1)(D+2)Y = \cos x$

$(D+1)\,\Psi = \cos x$

$(D+2)\,Y = \Psi$

$\Psi' + \Psi = \cos x$

$\Psi = e^{-x}\int e^{x}\cos x\,dx + ce^{-x}$

$= e^{-x}\left[\cos x\, e^{x} + \int \sin x\, e^{x} dx\right] + ce^{-x}$

$+ \sin x\, e^{x} - \int \cos x\, e^{x} dx$

$U = \int e^{x}\cos x\,dx$

$e^{-x}U + \cancel{ce^{-x}} = e^{-x}\left[\cos x e^{x} + \sin x e^{x} - U\right) + \cancel{ce^{-x}}$

$e^{-x}U = \cos x e^{x} + \sin x e^{x} - U$

$e^{-x}U + U = \cos x e^{x} + \sin x e^{x}$

$U(e^{-x}+1) = \cos x e^{x} + \sin x e^{x}$

$\int e^{x}\cos x\,dx = \dfrac{\cos x e^{x} + \sin x e^{x}}{e^{-x}+1}$

$\Psi = \dfrac{\cos x - \sin x}{e^{-x}+1} + ce^{-x}$

$Y' + 2Y = \dfrac{\cos x - \sin x}{e^{-x}+1} + ce^{-x}$

$Y_p = e^{-2x}\int \left[e^{2x}\left(\dfrac{\cos x - \sin x}{e^{-x}+1}\right) + ce^{-x}\right]dx$

$= e^{-2x}\left[\int \dfrac{e^{2x}\cos x}{e^{-x}+1}dx - \int \dfrac{e^{2x}\sin x}{e^{-x}+1}dx + \int ce^{-x}dx\right]$

etc

Answers to Integration Problems

1. $y = -(1+x)e^{-x} + c$

2. $y = x^2/2 \tan^{-1} + 1/2 \tan^{-1}x - x/2 + c$

3. $y = -x^2e^{-x} - 2(1+x)e^{-x} + c$

4. $y = x^3/3 \sin^{-1}x + x^2/3 (1-x^2)^{1/2} + 2/9(1-x^2)^{3/2}$

5. $y = -x/n \cos n x + 1/n^2 \sin n x + c$

6. $t = t^2/2 (\log t - 1/4) + c$

7. $z = 2z \sin z + 2 \cos z - z^2 \cos z + c$

8. $y = -e/3^{-3x} (x^2 + 2/3 x + 2/9) + c$

9. $y = (2-x^2) \cos x + 2x \sin x + c$

10. $y = 1/4 e^{2x} + 1/8 e^{2x} (\cos 2x + \sin 2x) + c$

11. $\log \cot |x|$

12. $\log |x-1| - \frac{1}{x-1} + c$

13. $\frac{-\sqrt{(a^2 - x^2)}}{a^2x} + c$

14. $\tan x - \sec x + c$

15. $2/3x^{3/2} + c$

16. $1/2 \log |1 + \sin 2t| + c$

17. $2/3x^{3/2} - x + 2x^{1/2} - 2 \log(1 + x^{1/2}) + c$

18. $1/15 (3x - 1)(2x + 1)^{3/2} + c$

19.

20. $\tan^{-1} (e^x)$

21. $\sin x - 1/3 \sin^3 x + c$

22. $1/a \log \left| \frac{x}{a + \sqrt{a^2 + x^2}} \right| + c$

23. $1/6 \log |(x+5)^5(x-1)| + c$

24. $\frac{x^{n+1}}{n+1} \log ax - \frac{x^{n+1}}{(n+1)^2} + c$

25. $x/2 [\sin(\log x) - \cos(\log x)] + c$

Test 35Tu27May59 CALCULUS III Name ______________________ BoxNo ____

Find the complete solution and check each by differentiation and substitution into the differential equation:

1. $y' = y$

2. $y' - 3y = 0$

3. $y' + 2y = 4$

4. $y' + 2y = x$

5. $y' + 2y = 5e^{3x}$

6. $y' + 2y = e^{-2x}$

7. Find the differential equation which has as its family of solutions the functions

$$y = x^2 + c\,e^{3x}$$

8. $y'' + 3y' + 2y = 4x$

9. $y'' + 3y' + 2y = 10 \sin x$

10. $y'' - 6y' + 9y = e^{3x}$

11. $y'' - x = \sin x$

12. $y'' - 6y' + 25y = 0$

MARKS
BOX 156

(CONT.)

$$Y = e^{-2x}\left[\int 4x e^{2x} dx - \int 4e^{2x}_{dx} + \int ce^{x}_{dx}\right] + c_2 e^{-2x}$$

$$= e^{-2x}\left[2xe^{2x} - 2e^{2x} - 2e^{2x} + ce^{x}\right] + c_2 e^{-2x}$$

$$Y = 2x - \cancel{4} + ce^{-x} + c_2 e^{-2x}$$

$$\bar{Y} = e^{-x}\int e^{x} 10\sin x\, dx + c_1 e^{-x}$$

$$U = \int e^{x} 10 \sin x\, dx$$

$$= e^{x} 10 \sin x - \int 10 \cos x\, e^{x} dx$$

$$= e^{x} 10 \sin x - 10 \cos x\, e^{x} - \int 10 \sin x\, e^{x} dx$$

$$2U = e^{x} 10 \sin x - 10 \cos x\, e^{x}$$

$$U = 5e^{x}(\sin x - \cos x)$$

$$\bar{Y} = 5(\sin x - \cos x) + c_1 e^{-x}$$

$$Y' + 2Y' = 5(\sin x - \cos x) + c_1 e^{-x}$$

$$Y = e^{-2x}\int e^{2x}\left[5(\sin x - \cos x) + c_1 e^{-x}\right]dx + c_2 e^{-2x}$$

$$\left[\int(5e^{2x}\sin x - 5e^{2x}\cos x + c_1 e^{x}\right]dx + c_2 e^{-2x}$$

$$= e^{-2x}\left[c_1 e^{x} + \int 5e^{2x}\sin x\, dx - \int 5e^{2x}\cos x\, dx\right] + c_2 e^{-2x}$$

$$\frac{U}{5} = \int e^{2x}\sin x\, dx$$

$$\frac{2U}{5} = \sin x\, e^{2x} + \int \cos x\, e^{2x}$$

$$+ \left(-\cos x \frac{e^{2x}}{2}\right) - \int \sin x \frac{e^{2x}}{2}$$

$$\frac{4U}{5} = 2\sin x\, e^{2x} - \cos x\, e^{2x} - U$$

$$\frac{9U}{5} = 2\sin x\, e^{2x} - \cos x\, e^{2x}$$

$$\rightarrow U = \frac{10\sin x\, e^{2x} - 5\cos x\, e^{2x}}{9}$$

$$\frac{V}{5} = \int e^{2x}\cos x\, dx$$

$$= \cos x \frac{e^{2x}}{2} + \int \sin x \frac{e^{2x}}{2}$$

$$\frac{2V}{5} = \cos x\, e^{2x} + \int \sin x\, e^{2x}$$

$$\frac{9V}{5} = 2\cos x\, e^{2x} + \sin x \frac{e^{2x}}{2} \quad \cancel{\int \cos x\, e^{2x}}$$

(OVER)

$$Y = e^{-2x}\left[C_1 e^{x} + \frac{10\sin x \text{ [illegible]}}{9}\right] + C$$

$$= C_1 e^{-x} + \frac{5\sin x - 15\cos x}{9} + C_2 e^{-2x}$$

$\sin x - 3\cos x$

82

Test 33Tu27May69 CALCULUS III Name MARKS BoxNo 156

Find the complete solmtion and check each by differentiation and substitution into the differential equation:

1. $y' = y$

$y' - y = 0$

$y = ce^{x}$ ✓

2. $y' - 3y = 0$

$y = ce^{3x}$

−1

3. $y' + 2y = 4$

$y_p = e^{-2x}\int 4e^{2x}dx$

$= e^{-2x}\, 2e^{2x}$

$= 2 \Rightarrow y = 2 + ce^{-2x}$

must not omit the x on e^{-2} one time might be excused but 3!!

4. $y' + 2y = x$

$y_p = e^{-2x}\int e^{2x}x\,dx$

$= e^{-2x}\left(\frac{xe^{2x}}{2} - \int \frac{e^{2x}}{2}dx\right)$

$= e^{-2x}\left(\frac{xe^{2x}}{2} - \frac{e^{2x}}{4}\right) \Rightarrow y = \frac{x}{2} - \frac{1}{4} + ce^{-2x}$

5. $y' + 2y = 5e^{3x}$

$e^{-2x}\int e^{2x}\, 5e^{3x}$

$\int 5e^{5x}$

$y_p = e^{-2x}e^{5x} = e^{3x} \Rightarrow y = e^{3x} + ce^{-2x}$ ✓

6. $y' + 2y = e^{-2x}$

$\int e^{2x}e^{-2x}$

$\int dx = x \Rightarrow y = xe^{-2x} + ce^{-2x}$

$= (x+c)e^{-2x}$ ✓

7. Find the differential equation which has as its family of solutions the functions

$y = x^2 + c\,e^{3x}$

$e^{3x}\int e^{-3x} f(x)\,dx = x^2$

$\int e^{-3x} f(x)\,dx = \frac{x^2}{e^{3x}}$

$y' + 3y = 2x\,e^{+3x}$

$y' - 3y = 2x - 3x^2$ ✓

$2x + \cancel{3ce^{3x}} - 3x^2 - \cancel{3ce^{3x}}$

$\cancel{0}$

8. $y'' + 3y' + 2y = 4x$

$(D^2+3D+2)Y = 4X$

$(D+1)(D+2)Y = 4X$

$(D+1)\bar{Y} = 4X$

$(D+2)Y = \bar{Y}$

$\bar{Y}' + \bar{Y} = 4X$

$\bar{Y} = e^{-x}\int e^{x}\, 4x\,dx + ce^{-x}$

$= e^{-x}(e^{x}4x - 4e^{x}) + ce^{-x}$

$= 4x - 4 + ce^{-x}$

$Y' + 2Y = 4x - 4 + ce^{-x}$

$Y = e^{-2x}\int e^{2x}(4x - 4 + ce^{-x})\,dx$

(CONT. ON OTHER SHEET)

-1

9. $y'' + 3y' + 2y = 10 \sin x$

$(D+1)(D+2)Y = 10 \sin X$

$(D+1)\bar{Y} = 10 \sin X$

$(D+2)Y = \bar{Y}$

$\bar{Y}' + \bar{Y} = 10 \sin x$

(CONT. ON OTHER SHEET)

-1

10. $y'' - 6y' + 9y = e^{3x}$

$(D^2 - 6D + 9)Y = e^{3x}$

$(D-3)^2 Y = e^{3x}$

$(D-3)\bar{Y} = e^{3x}$

$(D-3)Y = \bar{Y}$

$\bar{Y}' - 3\bar{Y}' = e^{3x}$

$\bar{Y} = e^{3x}\int \cancel{e^{-3x}e^{3x}} + ce^{3x}$

$= c_2 e^{3x}$

$Y' - 3Y' = c_2 e^{3x}$

$Y = e^{3x}\int \cancel{e^{-3x}}\, c_2 \cancel{e^{3x}} + c_3 e^{3x}$

$= e^{3x} c_2 x + c_3 e^{3x}$

$= e^{3x}(c_2 x + c_3) \frac{x^2}{2}$

-4

11. $y'' - x = \sin x$

$Y'' = \sin x + x$

$Y' = -\cos x + \frac{x^2}{2} + C_1$

$Y = -\sin x + \frac{x^3}{6} + C_1 X + C_2$ ✓

12. $y'' - 6y' + 25y = 0$

$(D^2 - 6D + 25)Y = 0$

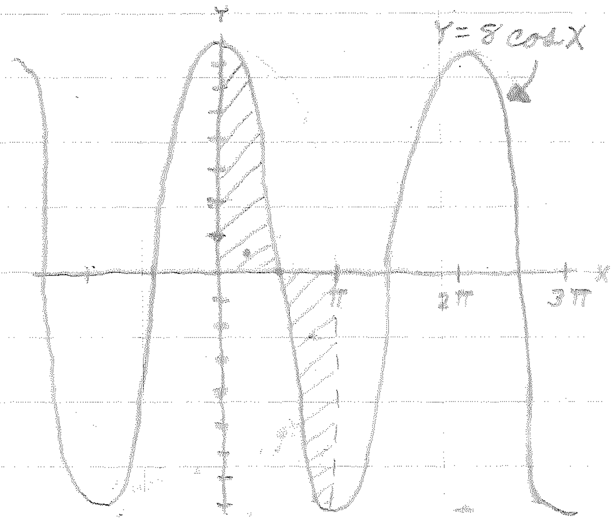
$r^2 - 6r + 25 = 0$

$(r-3)^2 + 16 = 0$

$y = e^{3x}(c_1 \cos 4x + c_2 \sin 4x)$

-9

5)



Find area bound by X axis of $Y = 8\cos X$ from $X = 0$ to $X = \pi$

$$A = 8\cos X \qquad \frac{A}{2} = 8\sin x \Big|_0^{\frac{\pi}{2}} = 8$$

$$A = 2\int_{\pi\,0}^{0\,\pi} 8\cos X \, dx \qquad A = 16$$

$$\int 8\cos X = {}^{\pm}8\sin X + C \qquad C = 0$$

$$\int_{\frac{\pi}{2}}^{0} 8\cos X = \left| {}^{-}8\sin 0 + 8\sin\frac{\pi}{2} \right| = 8$$

$$\int_{\pi}^{\frac{\pi}{2}} 8\cos X = \left| {}^{-}8\sin\frac{\pi}{2} + 8\sin\pi \right| = 8$$

$$\int_{\pi}^{0} 8\cos X = \int_{\frac{\pi}{2}}^{0} 8\cos X + \int_{\pi}^{\frac{\pi}{2}} 8\cos X = 16$$

BOB MARKS
2-14-69

35/50

1) $e^{\log 2} + \log e^2 =$
$\log 2 + 2\log e =$
$2 + \log 2$

−5

2) $Y = x \log x$
$Y' = x \frac{1}{x} + \log x$
$Y' = 1 + \log x$

3) $Y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$
$\log Y = \log(e^x - e^{-x}) - \log(e^x + e^{-x})$
$Y' = \left(\frac{e^x + e^{-x}}{e^x - e^{-x}} - \frac{e^x - e^{-x}}{e^x + e^{-x}}\right)\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)$
$= 1 - \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2$

4) $Y = \log \sin^2 x$
$Y' = \frac{2 \sin x \cos x}{\sin^2 x} = \frac{2 \cos x}{\sin x} = 2 \cot x$

5) $Y = e^{\sin x}$
$u = \sin x \quad \frac{du}{dx} = \cos x$
$\frac{dY}{dx} = \cos x \, e^{\sin x}$

6) $Y = 3^x$
$\log Y = x \log 3$
$\frac{Y'}{Y} = \log 3$
$Y' = 3^x \log 3$

7) $Y = \sec x \tan x + \log(\sec x + \tan x)$
$Y' = \sec^3 x + \sec x \tan^2 x + \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$

$= \sec x(\sec^2 x + \tan^2 x + 1 \frac{\tan x + \sec x}{\sec x + \tan x})$
$\sec x(\sec^2 x + \sec^2 x) = \sec x(2\sec^2 x)$
$\cancel{= \sec^3 x + \tan^2 x \sec x + \sec x}$
$Y' = 2 \sec^3 x$

8) $\log y = e^{-x}$

$\cancel{\frac{y'}{y} = -e^{-x}}$

$y = e^{e^{-x}}$

$u = e^{-x}$

$\cancel{\frac{du}{dx}} = -e^{-x}$

$y' = \cancel{-e^{-x}}\, e^{e^{-x}}$

$-1/0$

V_{oc}

Name Bob Marks

Box # 156 Seat 3-e

45

37/82 /80

1. $r = \frac{1}{3}\tan^3\theta - \tan\theta + \theta$

10

$$\frac{dr}{d\theta} = \frac{1}{3}(3\tan^2\theta\sec^2\theta) - \sec^2\theta + 1$$

$$= \tan^2\theta\sec^2\theta - \sec^2\theta + 1$$

$$= \sec^2\theta(\tan^2\theta - 1) + 1$$

$$= \tan^4\theta - 1 + 1$$
$$= (\tan^2\theta - 1)(\tan^2\theta + 1) - 1$$
$$\frac{dr}{d\theta} = \sec^2(\tan^2 - 1) + 1$$
$$= \sec^2\theta(\sec^2\theta - 2) + 1$$
$$= \sec^4\theta - 2\sec^2\theta + 1$$
$$= (\sec^2\theta - 1)^2$$
$$= (\tan^2\theta)^2$$
$$= \tan^4\theta$$

2. $y = \log\sqrt{\cos 2x}$

10

$$\frac{dy}{dx} = \frac{\frac{-\sin 2x}{(\cos 2x)^{\frac{1}{2}}}}{(\cos 2x)^{\frac{1}{2}}}$$

$$= \frac{-\sin 2x}{\cos 2x} = -\tan 2x$$

$dy = -\tan 2x\, dx$

3. $y' = \tan^2 x$

10

$$= \sec x^2 - 1$$

$$y = \tan x - x + C$$

$y = \tan x - x + C$

4. $y = \log\sqrt{\frac{1+\sin x}{1-\sin x}}$

10

$$= \frac{1}{2}\log(1+\sin x) - \frac{1}{2}\log(1-\sin x)$$

$$= \frac{\cos x}{2(1+\sin x)} + \frac{+\cos}{2(1-\sin x)} = \frac{\cancel{\cos x(1-\sin x)} + \cancel{\cos x(1+\sin x)}}{2\cos^2 x}$$

$y' = \sec x$

5. $s = e^t \cos t$

5

$$\ln s = t\ln e + \ln\cos t$$

$$\frac{\dot s}{s} = t' + \frac{\sin t}{\cos t} = t' - \tan t$$

$$\dot s = (e^t\cos t)(t - \tan t)$$

$$= te^t\cos t - e^t\sin t$$

$$= e^t\cancel{\cos t} + te^t\cos t - te^t\sin t - \cancel{e^t\cos t} - e^t\sin t$$

$$= e^t(t\cos t - t\sin t - \sin t)$$

Good try but not a very easy way in this case

$$\frac{d^2s}{dt^2} = te^t(\cos t - \sin t) - e^t\sin t$$

$\dot s = s(1-\tan t)$

$\ddot s = \dot s(1-\tan t) - s\sec^2 t$

Calculus II Name Boo MARKS
Box # 156 Seat 3-e

10 6. $y = \frac{1}{3}x^3\tan^{-1}x + \frac{1}{6}\log(x^2+1) - \frac{1}{6}x^2$ $y' = x^2\tan^{-1}x$

$= x^2\tan^{-1}x + \frac{x^3}{3(1+x^2)} + \frac{x}{3(x^2+1)} - \frac{x}{3}$

$= x^2\tan^{-1}x + \frac{x^3+x-(x^3+x)}{3(x^2+1)}$

$\tan^{-1}x$, $\frac{dx}{dt} = \sec^2 x$, $\frac{dt}{dx} = \cos^2 x$

5 7. Find the minimum and inflection points for $y = \frac{x}{\log x}$ $y' = 0$

Min: (e, e)
Infl: $\cancel{(e, e)}$

$y' = \frac{\log x - x\frac{1}{x}}{2\log x} = \frac{(\log x) - 1}{2\log x} = 0$

$\therefore \log x = 1 \Rightarrow x = e \Rightarrow y = e$

6 8. What is the minimum value of $y = ae^{kx} + be^{-kx}$? $y =$

$y = ae^{kx} + be^{-kx}$ $b = ae^{2kx}$

$y' = kae^{kx} - kbe^{-kx} = 0$ $\ln b = \ln a + 2kx$

$ae^{kx} - \frac{b}{e^{kx}} = 0$ $\frac{\ln b - \ln a}{2k} = x$

$ae^{kx} = \frac{b}{e^{kx}}$ $y = ae^{\frac{\ln b - \ln a}{2}} + be^{\frac{\ln a - \ln b}{2}}$

8 9. $\int \sqrt{4-9x^2}\,dx =$

$x = \frac{2}{3}\sin\theta$ $\frac{dx}{d\theta} = \frac{2}{3}\cos\theta$

$\int \sqrt{4 - 9\frac{4}{9}\sin^2\theta} = \sqrt{4-4\sin^2\theta}$

$y = \int 2\cos\theta\,dx$

$\theta = \sin^{-1}\left(\frac{3}{2}x\right)$

$y = \int 2\cos\theta\,dx$

$y = \int \frac{4}{3}\cos^2\theta\,d\theta$

$= \int \frac{2(1+\cos 2\theta)}{3}\,d\theta$

$= \int \frac{2}{3} + \frac{2}{3}\cos 2\theta$

$= \frac{2}{3}\theta + \frac{1}{3}\sin 2\theta$ $\frac{1}{3}\sin\left(2\sin^{-1}\frac{3x}{2}\right)$

$= \frac{2}{3}\sin^{-1}\left(\frac{3}{2}x\right) + \frac{1}{3}\sin\left(3\sin^{-1}x\right)$

10. $y = \log(\sec t + \tan t)$ $y' = \frac{\sec t\tan t + \sec^2 t}{\sec t + \tan t}$

$= \frac{\sec t(\tan t + \sec t)}{\sec t + \tan t}$

$= \sec t$

8 $\int \frac{dx}{\sqrt{x^2-4}} =$

$\int \frac{1}{2\tan\theta}\,dx$

$\int \frac{1}{2}\cot\theta\,dx$

$\int \sec\theta$

Let $x = 2\sec\theta$

$dx = 2\sec\theta\tan\theta\,d\theta$

$\int \sec\theta = \log(\sec t + \tan t) + C$

$=$ required $f(x)$

Name E. L. Godfrey

Box # ______ Seat Key

1. $r = \frac{1}{3}\tan^3\theta - \tan\theta + \theta$ $\qquad \frac{dr}{d\theta} = \tan^4\theta$

$\frac{dr}{d\theta} = \tan^2\theta\sec^2\theta - \underbrace{\sec^2\theta + 1}_{-(1+\tan^2\theta)}$

$= \tan^2\theta(\sec^2\theta - 1) = \tan^4\theta$

2. $y = \log\sqrt{\cos 2x}$ $\qquad dy = -\tan 2x\, dx$

$= \frac{1}{2}\log\cos 2x$

$y' = -\frac{\sin 2x}{\cos 2x} = -\tan 2x$

3. $y' = \tan^2 x$ $\qquad y = \tan x - x + C$

$= \sec^2 x - 1$

4. $y = \log\sqrt{\frac{1+\sin x}{1-\sin x}}$ $\qquad y' = \frac{1}{\cos x} = \sec x$

$2y = \log(1+\sin x) - \log(1-\sin x)$

$2y' = \frac{\cos x}{1+\sin x} + \frac{\cos x}{1-\sin x} = \frac{2\cos x}{1-\sin^2 x} = \frac{2\cos x}{\cos^2 x}$

5. $s = e^t\cos t$ $\qquad \frac{d^2s}{dt^2} = -2e^t\sin t$

$\frac{ds}{dt} = e^t\cos t - e^t\sin t$

$\frac{d^2s}{dt^2} = e^t\cos t - e^t\sin t - e^t\sin t - e^t\cos t$

6. $y = \frac{1}{3}x^3\tan^{-1}x + \frac{1}{6}\log(x^2+1) - \frac{1}{6}x^2$ $\qquad y' = x^2\tan^{-1}x$

$$y' = x^2\tan^{-1}x + \frac{x^3}{3}\frac{1}{1+x^2} + \frac{x}{3(1+x^2)} - \frac{x}{3}$$

$$\frac{1}{3}\frac{x(x^2+1)}{1+x^2} - \frac{x}{3} = 0$$

7. Find the minimum and inflection points for $\qquad$ Min: (e, e)

$y = \frac{x}{\log x}$, $y' = \frac{\log x - 1}{(\log x)^2} = 0$, $\log x = 1$, $x = e, y = e$ $\qquad$ Infl: $\left(e^2, \frac{e^2}{2}\right)$

$$y'' = \frac{(\log x)^2\frac{1}{x} - \frac{2}{x}\log x(\log x - 1)}{(\log x)^2} = \frac{\frac{2}{x} - \frac{1}{x}\log x}{\log x}$$

$x = e, y'' = \frac{2}{e} - \frac{1}{e} = \frac{1}{e} > 0$, min

$y'' = 0$, $\log x \neq 0$

$\frac{2}{x} = \frac{1}{x}\log x$, $\log x = 2$, $x = e^2$, $y = \frac{e^2}{2}$

8. What is the minimum value of $y = ae^{kx} + be^{-kx}$? $\qquad y = 2\sqrt{ab}$

$y' = ake^{kx} - bke^{-kx} = 0$ $\qquad y'' = ak^2e^{kx} + bk^2e^{-kx} = k^2y$

$ake^{kx} = bke^{-kx}$ $\qquad = 2k^2\sqrt{ab} > 0$

$e^{2kx} = \frac{b}{a}$, $e^{kx} = \sqrt{\frac{b}{a}}$ $\qquad \therefore y = \min$

$y = a\sqrt{\frac{b}{a}} + b\sqrt{\frac{a}{b}} = 2\sqrt{ab}$ when $x = \frac{\log b - \log a}{2k}$

9. $\int\sqrt{4-9x^2}\,dx = \int\frac{4}{3}\cos^2\theta\,d\theta = \frac{2}{3}\int(1+\cos 2\theta)\,d\theta$

Let $3x = 2\sin\theta$ $\qquad = \frac{2}{3}\theta + \frac{1}{3}\sin 2\theta + C$

$dx = \frac{2}{3}\cos\theta\,d\theta$ $\qquad = \frac{2}{3}\sin^{-1}\frac{3x}{2} + \frac{2}{3}\,\frac{3x}{2}\,\frac{\sqrt{4-9x^2}}{2} + C$

$\qquad = \frac{2}{3}\sin^{-1}\frac{3x}{2} + \frac{x}{2}\sqrt{4-9x^2} + C$

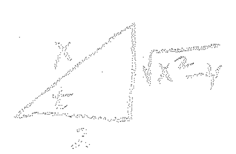
10. $y = \log(\sec t + \tan t)$ $\qquad y' = \frac{\sec t(\tan t + \sec t)}{\sec t + \tan t} = \sec t$

$$\int\frac{dx}{\sqrt{x^2-4}} = \int\frac{2\sec t\tan t}{2\tan t}\,dt = \int\sec t\,dt = \log(\sec t + \tan t) + C'$$

Let $x = 2\sec t$ $\qquad = \log\frac{x+\sqrt{x^2-4}}{2} + C'$

$dx = 2\sec t\tan t\,dt$ $\qquad = \log(x+\sqrt{x^2-4}) + C$

Note $C = C' - \log 2$



90 Calculus II — Box No 15G

Name Bob Marks

1. Given $y = \sqrt{x}$, solve for x and find $\frac{dx}{dy}$ and then get $\frac{dy}{dx}$

$Y = X^{\frac{1}{2}}$

$X = Y^2$

$\frac{dX}{dY} = 2Y$

$\frac{dY}{dX} = \frac{1}{2Y}$

$\frac{dY}{dX} = \frac{1}{2X^{\frac{1}{2}}} = \frac{1}{2\sqrt{X}}$

2. Since $\sin x = \cos x \tan x$ if $y = \tan x$ find y' using only derivatives of sin x and cos x

$Y = \tan X$

$Y = \frac{\sin X}{\cos X}$

$\dot{Y} = \frac{\cos^2 X + \sin^2 X}{\cos^2 X} = \frac{1}{\cos^2 X}$

$\dot{Y} = \sec^2 X$

3. Given $x = 16t^2 \sin A = \frac{5}{\cos A}$ find the angle A that minimizes time t.

(BACK)

$\cancel{t^2 = \frac{5}{16 \sin A \cos A}}$

$\cancel{t = \sqrt{\frac{5}{16 \sin A \cos A}}} = \frac{(5)^{\frac{1}{2}}}{4(\sin A \cos A)^{\frac{1}{2}}}$

$\cancel{= \frac{5^{\frac{1}{2}}}{4}(\sec A \csc A)^{\frac{1}{2}}}$

$= \frac{5^{\frac{1}{2}}}{4}(\sec A \tan A \cot A \csc A)^{\frac{1}{2}}$

4. $y = 5 \sin^4 3t^2$ $\quad \dot{y} = 240t \sin^2 3t^2 \cos^2 3t^2$

$U = 3t^2$ $\quad$ $\cancel{Y = 5 \sin^4 U}$

$\frac{du}{dx} = 6t$ $\quad$ $\cancel{\frac{dy}{du} = 10 \sin^2 U \cos^2 U} = 40 (\sin U \cos U)(\sin U \cos U)$

$y = \sin x \cos x$, Find y' two ways

$\dot{y} = \sin X \sin X + \cos^2 X$

$= \cos^2 X - \sin^2 X$

$Y = \tan X \cos^2 X$

$\dot{Y} = \tan X \cos^2 X + \sec^2 X \cos^2 X$

$= \tan X (2 \sin X \cos X)$

$\neq 2 \sin^2 X$

$$X = 16t^2 \sin A = \frac{5}{\cos A}$$

$$t^2 = \frac{5}{16 \sin A \cos A}$$

$$t = \frac{\sqrt{5}}{4(\sin^{\frac{1}{2}} A \cos^{\frac{1}{2}} A)}$$

$$t = \frac{\sqrt{5}}{4}(\sec^{\frac{1}{2}} A \csc^{\frac{1}{2}} A)$$

$$\csc^{\frac{1}{2}} A \cot^{\frac{1}{2}} A \sec^{\frac{1}{2}} A$$

$$\frac{dt}{dA} = \frac{\sqrt{5}}{4}\left(\cancel{\csc^{\frac{1}{2}} A \sec^{\frac{1}{2}} A \tan^{\frac{1}{2}} A - \sec^{\frac{1}{2}} A \tan^{\frac{1}{2}} A \cot A}\right)$$

$$\cancel{\sec^{\frac{1}{2}} A \cot A + \sec^{\frac{1}{2}} A}$$

$$\cancel{\csc^{\frac{1}{2}}} A \cancel{\sec^{\frac{1}{2}}} A \tan^{\frac{1}{2}} A = \cancel{\csc^{\frac{1}{2}}} A \cot^{\frac{1}{2}} A \cancel{\sec^{\frac{1}{2}} A}$$

$$\tan^{\frac{1}{2}} A = \cot^{\frac{1}{2}} A$$

$$\tan A = \cot A$$

$$A = 45^\circ, 225^\circ, 405^\circ, \text{etc}$$

Set up completely and evaluate the definite integral for the volume shown in Fig. 1 if the cross-sections perpendicular to the z-axis are right triangles.

$$A(h) = \frac{1}{2}BH \Rightarrow A(z) = \frac{1}{2}xy = \frac{1}{2}\left(\frac{16-z^2}{8}\right)\left(\frac{z^2+8}{8}\right)$$

$$dV = A(z)dz = \frac{1}{128}\left[128 + 8z^2 - z^4\right]dz$$

$$V = \frac{1}{128}\int_0^4 (128 + 8z^2 - z^4)\,dz$$

$$= \frac{1}{128}\left[\overset{2\cdot 4^3}{128z} + \frac{8}{3}z^3 - \frac{1}{5}z^5\right]_0^4$$

$$= \frac{1}{128}\left[2\cdot 4^4 + \frac{2}{3}4^4 - \frac{4}{5}4^4\right] = \frac{2\cdot 4^4}{2\cdot 4^3}\left[1 + \frac{1}{3} - \frac{2}{5}\right]$$

$$= 4\left(\frac{15+5-6}{15}\right) = \frac{4\cdot 14}{15} = \underline{\underline{\frac{56}{15}}}\text{ cu. units}$$

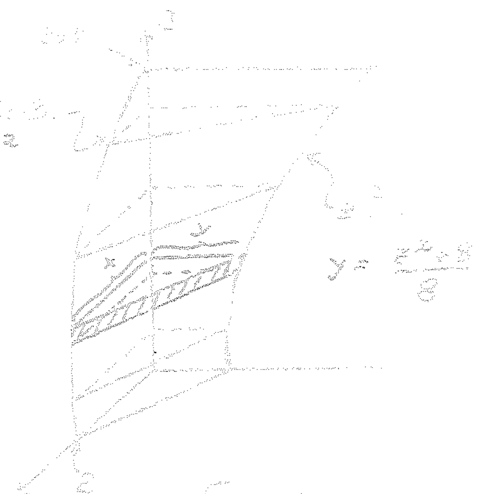


Fig. 1

Set up completely, BUT DO NOT EVALUATE the definite integral for the volume of the solid of revolution generated by revolving about the line x = -1 the area enclosed by the lines y = 0 and y = 2 and by the parabolas $y^2 = 4x$ and $y^2 = 8(x-1)$. [See Fig. 2] Be sure to draw in a typical volume element and label any points and distances used in setting up the integral.

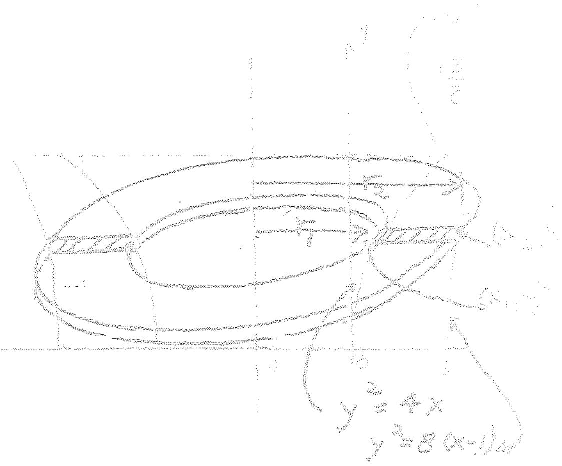


Fig. 2

$$A(h) = \pi r_2^2 - \pi r_1^2$$

$$A(y) = \pi\left[(x_1+1)^2 - (x_2+1)^2\right] = \pi\left[\left(\frac{y^2}{4}+1\right)^2 - \left(\frac{y^2+8}{8}+1\right)^2\right]$$

$$= \pi\left[\frac{y^4 + 8y^2 + 16}{16} - \frac{y^4 + 32y^2 + 256}{64}\right] =$$

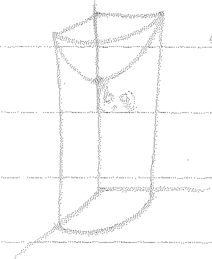
$$= \frac{\pi}{64}\left[3y^4 - 192\right],\quad 0 \le y \le 2$$

$$dV = \frac{\pi}{64}\left[3y^4 - 192\right]dy$$

$$\therefore V = \frac{\pi}{64}\int_0^2 (3y^4 - 192)\,dy.$$

1) $dV = \frac{1}{2} XY\,dz$

$$V = \int_0^4 \frac{1}{2}\left(\frac{16-z^2}{8}\right)\left(\frac{z^2+8}{8}\right)dz$$
$$= \frac{1}{128}\int_0^4 (16-z^2)(z^2+8)\,dz$$
$$= \frac{1}{128}\int_0^4 (16z^2+128-z^4-8z^2)\,dz$$
$$= \frac{1}{128}\int_0^4 (-z^4+8z^2+128)\,dz$$
$$= \frac{1}{128}\left[-\frac{z^5}{5}+\frac{8}{3}z^3+128z\right]_0^4$$
$$= \frac{1}{128}\left[-\frac{1024}{5}+\frac{512}{3}+512\right]$$
$$= \frac{1}{128}\left[\frac{7168}{15}\right]$$
$$= \frac{56}{15} \text{ cubic units}$$

446-1) 

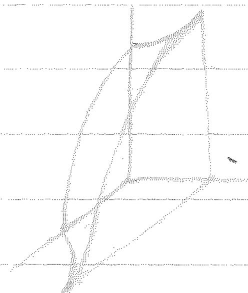
a) $V_1 = \int_0^3 \pi y^2 dx$

$= \int_0^3 \pi (9-x^2)^2 dx$

$= \pi \int_0^3 (81 - 18x^2 - x^4) dx$

$= \pi \left(81x - 6x^3 - \frac{x^5}{5}\right)\Big|_0^3$

$= \frac{648\pi}{5}$



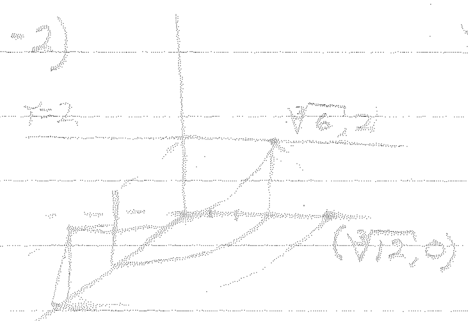
b) $x^2 = 9 - y$

$V_2 = \int_0^9 \pi x^2 dy$

$= \pi \int_0^9 (9-y) dy$

$= \pi \left[9y - \frac{y^2}{2}\right]_0^9$

$= \frac{81\pi}{2}$

446-2) 

$y = \frac{12 - x^3}{3}$

$V_1 = \int_0^{\sqrt[3]{6}} \pi y^2 dx$

$= \int_0^{\sqrt[3]{6}} \pi 4 \, dx$

$= 4\pi x \Big|_0^{\sqrt[3]{6}}$

$= 4\pi \sqrt[3]{6}$

$V_2 = \int_0^{\sqrt[3]{12}} \pi y^2 dx$

$= \int_0^{\sqrt[3]{12}} \pi \left(\frac{12 - x^3}{3}\right)^2 dx$

$= \int_0^{\sqrt[3]{12}} \pi \left(\frac{144 - 24x^3 + x^9}{9}\right) dx$

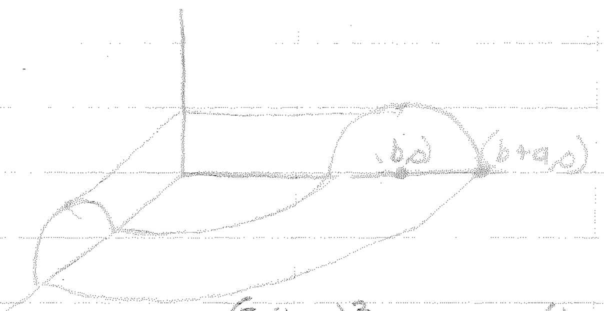
$= \pi \left[\frac{144}{9}x - \frac{2x^4}{3} + \frac{x^{10}}{10}\right]_0^{\sqrt[3]{12}}$

$= \pi \left(16\sqrt[3]{12} - \frac{24\sqrt[3]{12}}{3} + \frac{36\sqrt[3]{12}}{10}\right)$

$= \sqrt[3]{12}\,\pi \left(16 - 8 + \frac{18}{5}\right)$

$= \frac{\sqrt[3]{12}\,\pi\, 58}{5}$

446-3)



$$(x-b)^2+y^2=a^2$$

$$V=\int_{b-a}^{b+a} 2\pi x y\,dx$$

$$=2\pi\int_{b-a}^{b+a} x\left[a^2-x^2+2bx-b^2\right]dx$$

$$=2\pi\int_{b-a}^{b+a}\left(a^2x-x^3+2bx^2-b^2x\right)dx$$

$$=2\pi\left[\frac{a^2x^2}{2}-\frac{x^4}{4}-\frac{2bx^3}{3}-\frac{b^2x^2}{2}\right]_{b-a}^{b+a}$$

$$=2\pi\left[\frac{x^2}{2}(a^2-b^2)-\frac{x^4}{4}-\frac{2bx^3}{3}\right]_{b-a}^{b+a}$$

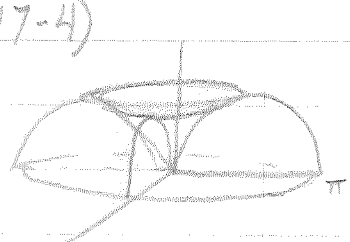
$$=2\pi\left\{\left[\frac{(b+a)^2}{2}(a^2-b^2)-\frac{(b+a)^4}{4}-\frac{2b(b+a)^3}{3}\right]-\left[\frac{(b-a)^2}{2}(a^2-b^2)-\frac{(b-a)^4}{4}-\frac{2b(b-a)^3}{3}\right]\right\}$$

$$=2\pi\left\{\frac{b^2}{2}(a^2-b^2)+ab(a^2-b^2)+\frac{a^2(a^2-b^2)}{2}-\cancel{\frac{b^4}{4}}-\frac{4b^3a}{4}-\cancel{\frac{6b^2a^2}{4}}-\frac{4ba^3}{4}-\cancel{\frac{a^4}{4}}\right.$$
$$-\cancel{\frac{2b(b)^3}{3}}-\frac{2b3b^2a}{3}-\cancel{\frac{2b3a^2b}{3}}-\frac{2ba^3}{3}-\frac{(a^2-b^2)b^2}{2}+\frac{(a^2-b^2)2ab}{2}-\frac{(a^2-b^2)b^2}{2}$$
$$+\cancel{\frac{b^4}{4}}-\cancel{\frac{4b^3a}{4}}+\cancel{\frac{6b^2a^2}{4}}-\cancel{\frac{4ba^3}{4}}+\cancel{\frac{a^4}{4}}+\cancel{\frac{2b\,b^3}{3}}-\frac{2b3b^2a}{3}+\cancel{\frac{2b3ba^2}{3}}$$
$$\left.-\frac{2ba^3}{3}\right\}$$

$$=2\pi\left(\cancel{\frac{b^2a^2}{2}}-\cancel{\frac{b^4}{2}}+\cancel{a^3b}-\cancel{ab^3}+\frac{a^4}{2}-\cancel{\frac{a^2b^2}{2}}-\cancel{b^3a}-\cancel{ba^3}-2ab^3-\frac{2ba^3}{3}\right.$$
$$\left.-\cancel{\frac{a^2b^2}{2}}+\cancel{\frac{b^4}{2}}+\cancel{a^3b}-\cancel{ab^3}-\frac{a^2b^2}{2}+\cancel{\frac{b^4}{2}}-b^3a-\cancel{ba^3}-2b^3a-\cancel{\frac{2ba^3}{3}}\right)$$

$$=\left(-\frac{4ba^3}{3}-b^2a^2-5ab^3+\frac{b^4}{2}+\frac{a^4}{2}\right)2\pi$$

447-4)



$$V=\int_0^{\pi}\pi y^2\,dx$$

$$=\int_0^{\pi}\pi(\sin^2x)\,dx$$

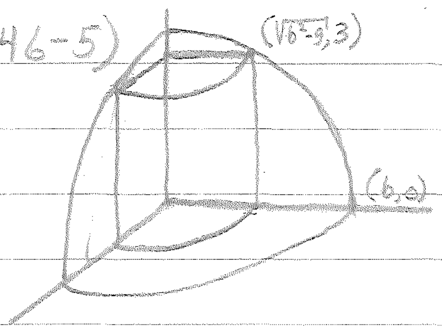
$$=\pi\left[\frac{1}{2}x-\frac{1}{4}\sin 2x\right]_0^{\pi}$$

$$=\pi$$

$$V=\pi\left(\left(\frac{\pi}{2}\right)-\frac{1}{4}\right)-(0)$$

$$=\frac{2\pi^2-1}{4}$$

446-5)



$$x^2+Y^2=b^2$$

$$X=\sqrt{b^2-9}$$

$$V_1=\int_0^b \pi Y^2\,dx$$
$$=\int_0^b \pi(b^2-x^2)\,dx$$
$$=\pi\left(b^2x-\frac{x^3}{3}\right)\Big|_0^b$$
$$=b^3\pi-\frac{b^3}{3}$$
$$V_1=\frac{2}{3}\pi b^3$$

$$V_2=\int_0^3 \pi X^2\,dY$$
$$=\int_0^3 \pi(b^2-9)\,dY$$
$$=\pi(b^2-9)\,Y\Big|_0^3$$
$$=3\pi b^2-27\pi$$

$$V_3=\int_3^b \pi X^2\,dY$$
$$=\int_3^b \pi(b^2-x^2)\,dx$$
$$=\pi\left[\left(b^3-\frac{b^3}{3}\right)-(3b^2-9)\right]$$
$$=\pi\left(\frac{2}{3}b^3-3b^2+9\right)$$

$$V=V_1-V_2-V_3=\frac{2}{3}\pi b^3-3\pi b^2+27\pi-\cancel{\frac{2b^3\pi}{3}}+3\pi b^2+9\pi$$
$$=36\pi$$

447-8) $V = \int_0^{a+b} \pi y^2 dx$

$$= \int_0^{a+b} \left(a^2 - (x^2 - b^2)\right)^2 dx$$

$$= \int_0^{a+b} \left(a^2 - x^2 + 2bx - b^2\right) dx$$

$$= \left(a^2x - \frac{x^3}{3} + \frac{\cancel{2}bx^2}{\cancel{2}} - b^2x\right)\Big|_0^{a+b}$$

$$= a^2(a+b) - \frac{(a+b)^3}{3} + b(a+b)^2 - b^2(a+b)$$

$$= \cancel{a^3} + a^2b - \frac{a^3}{3} - \frac{3a^2b}{3} - \frac{3ab^2}{3} - \frac{b^3}{3} + ba^2 + 2ab^2 + \cancel{b^3} - b^2a - \cancel{b^3}$$

$$= \frac{2a^3}{3} - \frac{7b^3}{3} - 2a^2b - 5b^2a$$

Bob Marks 4

2 Th 3

431-7a) $\int_1^3 x^2 dx = \frac{x^3}{3}\Big|_1^3 = 9 - \frac{1}{3} = \frac{26}{3}$

b) $\int_0^1 x e^{x^2} dx = \frac{1}{2} e^{x^2}\Big|_0^1 = \frac{e-1}{2}$

c) $\int_0^1 \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x \Big|_0^1 = \frac{\pi}{2}$

d) $\int_{\pi/2}^{\pi} \sin x \, dx = -\cos x \Big|_{\pi/2}^{\pi} = 1$

e) $\int_0^{\pi/2} \cos x \, dx = \sin x \Big|_0^{\pi/2} = 1$

f) $\int_0^{\pi/2} \sin^2 x \, dx$

$\int_0^{\pi/2} \left(\frac{1}{2} - \frac{\cos 2x}{2}\right) dx$

$\left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^{\pi/2} = \frac{\pi}{4}$

g) $\int_0^1 Y \, dx \qquad Y = x^3$

$\int_0^1 x^3 dx = \frac{x^4}{4}\Big|_0^1 = \frac{1}{4}$

h) $\int_{-2}^{3} \cos x \, dx = \sin x \Big|_{-2}^{3} = \sin 3 - \sin(-2)$

FINAL) 3-d) $\frac{dY}{dx} = \frac{x}{\sqrt{1+x}} \qquad Y = \int \frac{U-1}{\sqrt{U}} du$

$U = 1 + x \qquad U_1 = U - 1 \qquad dV = U^{-\frac{1}{2}}$

$x = U - 1 \qquad dU_1 = dU \qquad V = 2U^{\frac{1}{2}}$

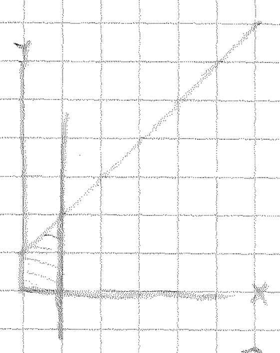
$dU = dx$

$Y = (2U^{\frac{1}{2}})(U-1) - \int 2U^{\frac{1}{2}} dU$

$= 2U^{\frac{3}{2}} - 2U^{\frac{1}{2}} - \frac{4}{3}U^{\frac{3}{2}} = -2U^{\frac{1}{2}} + \frac{2}{3}U^{\frac{3}{2}}$

$= \frac{2}{3}(x+1)^{\frac{3}{2}} - 2\sqrt{1+x} + C$

FINAL-9)



$S_n = \Delta x(\Delta x + 1) + \Delta x(2\Delta x + 1) \ldots \Delta x(n\Delta x + 1)$

$= (\Delta x^2 + \Delta x) + (2\Delta x^2 + \Delta x) \ldots n\Delta x^2 + \Delta x$

$= n\Delta x + \Delta x^2 (1 + 2 + 3 \ldots n)$

$= n\Delta x + \Delta x^2 \left(\frac{n^2+n}{2}\right)$

$n\Delta x = 1$

$\Delta x = \frac{1}{n} \qquad = 1 + \frac{n^2+n}{2n^2} = 1 + \frac{n^2}{2n^2} + \frac{n}{2n^2} \Rightarrow \lim_{n\to\infty} S_n = \frac{3}{2}$

$A = \int_0^1 (x+1) dx$

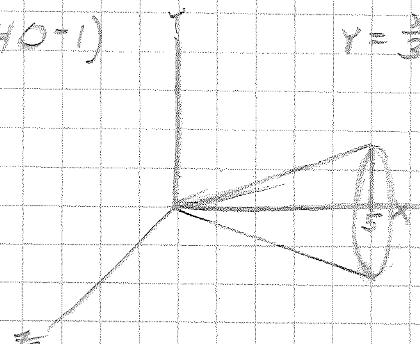
$= \left[\frac{x^2}{2} + x\right]_0^1$

$= \frac{3}{2}$

Bob Marks
3F4

6

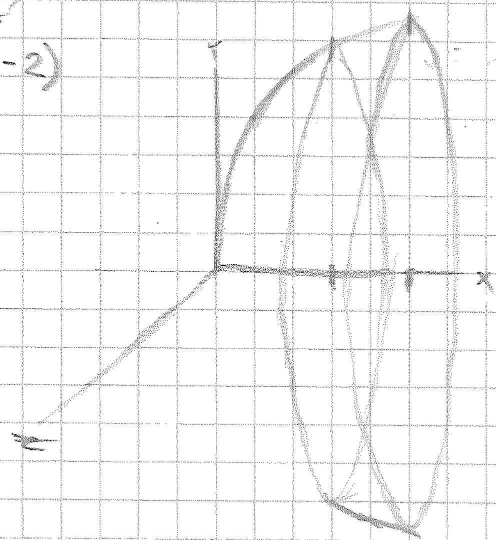
440-1)



$Y = \frac{x}{3}$

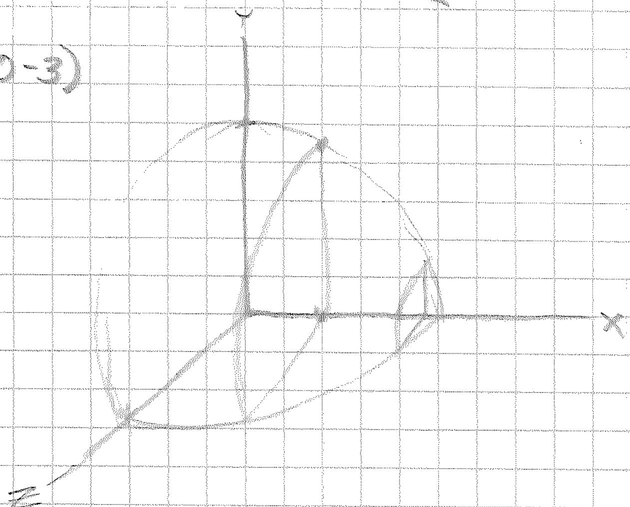
$$V = \int_0^5 \pi Y^2 \, dx$$
$$= \pi \int_0^5 \frac{x^2}{9} \, dx$$
$$= \pi \frac{1}{27} x^3 \Big|_5^0$$
$$= \frac{125\pi}{27}$$

440-2)



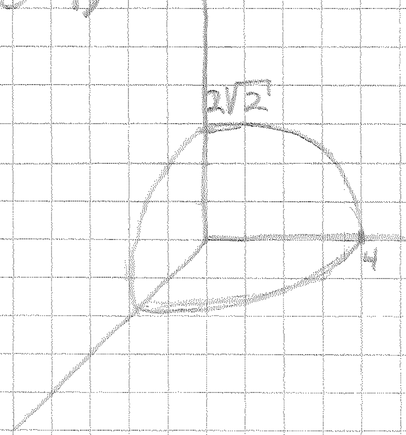
$$Y^2 = 16X$$
$$V = \int_3^5 \pi Y^2 \, dx$$
$$= \int_3^5 16\pi x \, dx$$
$$= 8\pi x^2 \Big|_3^5$$
$$= 200\pi - 72\pi = 128\pi$$

440-3)



$$x^2 + Y^2 = 25$$
$$V = \int_2^4 \pi Y^2 \, dx$$
$$= \pi \int_2^4 (25 - x^2) \, dx$$
$$= \pi \left[25x - \frac{x^3}{3} \right]_2^4$$
$$= \pi\left(400 - \frac{64}{3}\right) - \pi\left(50 - \frac{8}{3}\right)$$
$$= \pi\left(\frac{1136}{3} - \frac{142}{3}\right)$$
$$= \frac{994\pi}{3}$$

440-4)



$3x^2+6y^2=48$

$\frac{x^2}{16}+\frac{y^2}{8}=1$

$\frac{1}{2}V=\int_4^0 \pi y^2\,dx$

$=\pi\int_4^0\left(\frac{48-3x^2}{6}\right)dx$

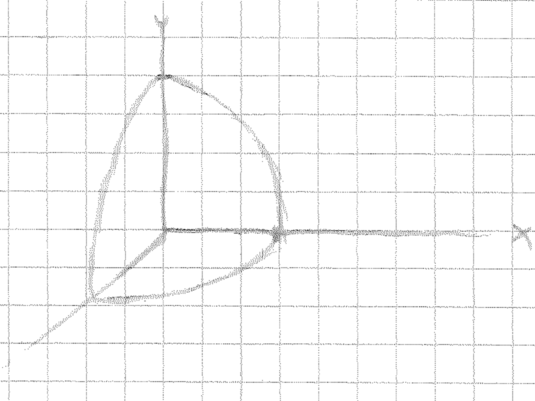
$=\pi\int_4^0\left(8-\frac{x^2}{2}\right)dx$

$=\pi\left[8x-\frac{x^3}{6}\right]_4^0$

$=\pi\left(32-\frac{32}{3}\right)$

$=\pi\,\frac{64}{3}\Rightarrow V=\frac{128\pi}{3}$ ✓

440-5)



$6x^2+3y^2=48$

$\frac{y^2}{16}+\frac{x^2}{8}=1$

$\frac{1}{2}V=\int_0^{2\sqrt{2}}\pi y^2\,dx$

$=\pi\int_0^{2\sqrt{2}}\left(\frac{48-6x^2}{3}\right)dx$

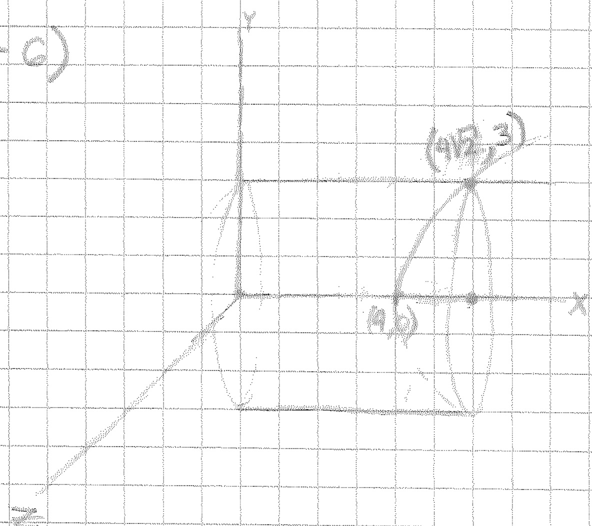
$=\pi\int_0^{2\sqrt{2}}(16-2x^2)\,dx$

$=\pi\left[16x-\frac{2}{3}x^3\right]_0^{2\sqrt{2}}$

$=\pi\left(32\sqrt{2}-\frac{32\sqrt{2}}{3}\right)$

$=\frac{64\sqrt{2}\,\pi}{3}\Rightarrow V=\frac{128\sqrt{2}\,\pi}{3}$ ✓

440-6)



$9x^2-16y^2=144$

$\frac{x^2}{16}-\frac{y^2}{9}=1$

$V_1=\int_4^{4\sqrt{2}}\pi y^2\,dx$

$=\pi\int_4^{4\sqrt{2}}\left(\frac{9x^2-144}{16}\right)dx$

$=\pi\int_4^{4\sqrt{2}}\left(\frac{9}{16}x^2-9\right)dx$

$=\pi\left[\frac{3}{16}x^3-9x\right]_4^{4\sqrt{2}}$

$=\pi\left[\left(\frac{3}{16}(128)\sqrt{2}-36\sqrt{2}\right)-\left(\frac{192}{16}-36\right)\right]$

$V_2=\int_0^{4\sqrt{2}}\pi y^2$

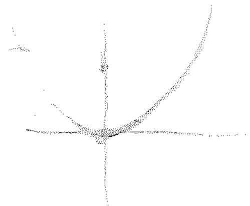
$=\int_0^{4\sqrt{2}}\pi 9$

$=36\pi\sqrt{2}$

$V_2-V_1=V_T=24\pi(2\sqrt{2}-1)$ ✓

111 Name BOB MARKS 70 Box No. 156

1. Find the radius of curvature at the vertex of the parabola $x^2 = 4ay$.



-15

Consider the arc of the curve $x = y^2$ from $x = 0$ to $x = 2$:

2. Find the surface area generated by revolving the arc around the x-axis.

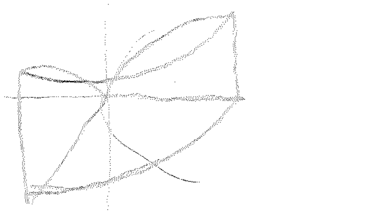
$A = \int_0^2 2\pi \cancel{y}\, \cancel{dx}\, ds$

$Y = \sqrt{x}$

$= \int_0^2 2\pi (x)^{\frac{1}{2}} dx$

$= \frac{\cancel{2}\pi}{\cancel{2}\sqrt{x}} \Big|_0^2$ -15

$= \frac{\pi}{\sqrt{2}}$



3. Find the volume generated by rotating about the x-axis the area bounded by the arc, the line $x = 2$ and the x-axis.

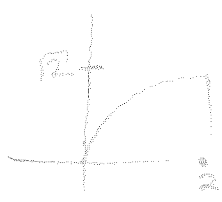
$V = \int_0^2 \pi Y^2 dx$

$= \int_0^2 \pi x\, dx$

$= \pi \frac{x^2}{2} \Big|_0^2$

$= 2\pi$ ✓

4. Set up the integral for the length of the arc. Integrate it if you have time later.

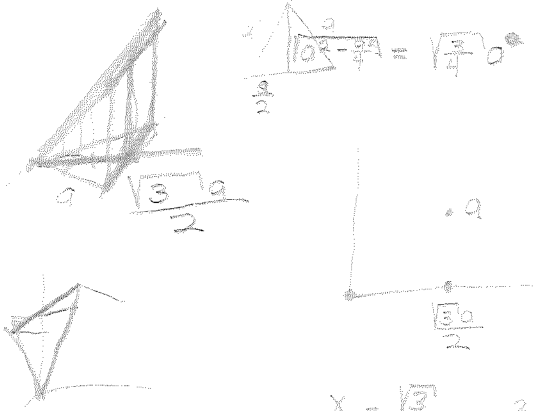


$Y = \sqrt{x}$

$\frac{dY}{dx} = \frac{1}{2\sqrt{x}}$

$S = \int_0^2 \sqrt{1 + \frac{1}{4x}}\, dx$ ✓

5. The base of a certain solid is an equilateral triangle of side a with one vertex at the origin and an altitude along the x-axis. Each plane section perpendicular to the x-axis is a square, one side of which lies in the base of the Solid. Find the volume of the solid.



$\sqrt{a^2 - \frac{a^2}{4}} = \sqrt{\frac{3}{4}}\, a$

$V = \int Y^2 dx$

$= \int_0^{\frac{\sqrt{3}a}{2}} \frac{4}{3} x^2 dx$

$= \frac{4x^3}{9} \Big|_0^{\frac{\sqrt{3}a}{2}}$

$= \frac{4(3)\sqrt{3}\, a^3}{3(8)2} = \frac{\sqrt{3}\, a^3}{6}$ ✓

$\frac{x}{y} = \frac{\sqrt{3}}{2}$ $\quad \frac{2}{\sqrt{3}} x = Y$

Find the radius of curvature at the vertex of the parabola $x^2 = 4ay$.

$2x = 4ay'$ at vertex $(0,0)$, $y' = 0$

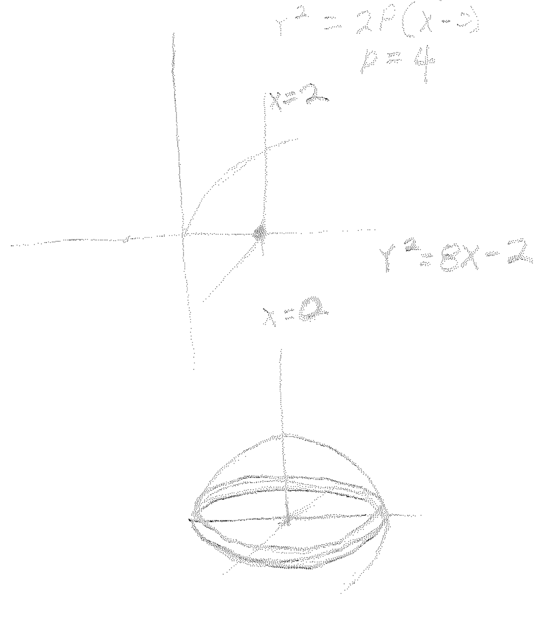
$y = \sqrt{x}$, $y' = \frac{1}{2\sqrt{x}}$

25

15F25 April 1969 CALCULUS III Name BOB MARKS Box No. 156

-1. 1. Find the volume generated by revolving the area bounded by the parabola $y^2 = 8x$ and its latus rectum about the latus rectum. Use the shell method.

5

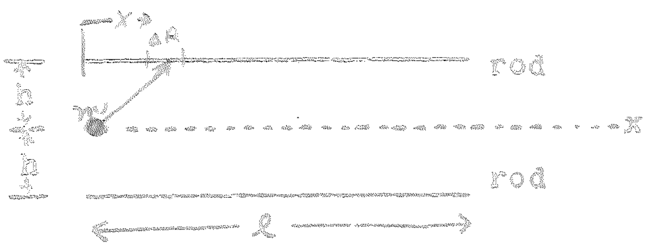


$$dV = \pi X^2 dY$$
$$V = \pi \int_0^4 \frac{Y^4}{64} dY$$
$$= \pi \left. \frac{Y^5}{320} \right|_0^4$$
$$= \pi \frac{1024}{320}$$
$$= \frac{512\pi}{160} = \frac{256\pi}{80} = \frac{84\pi}{20} = \frac{21\pi}{10} \text{ cu. units}$$

64
16
384
64
1024

2. Find the force exerted on the mass m by the two parallel rods as shown in the figure. Each rod has a uniformly distributed mass M.

5



$$\frac{GmM}{r^2}$$

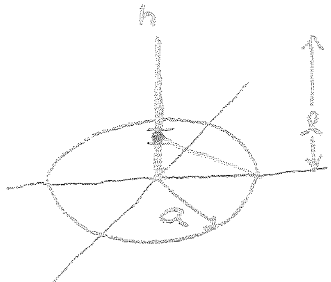
$$\Delta F = \frac{GmM}{[(\Delta x)^2 + h^2]}$$

$$F = \int_0^{\ell} \frac{GMm}{(dx)^2 + h^2}$$

15P25 April 1969 Calculus III Name MARKS BoxNo. 156

3. Find the attraction of a solid disc of mass Mt per unit of surface area on a rod of uniformly distributed mass m and length ℓ normal to the center of the disc. Note: The attraction of the disc for a point mass is

$$2\pi G m M t\left(1 - \frac{h}{\sqrt{a^2 + h^2}}\right)$$



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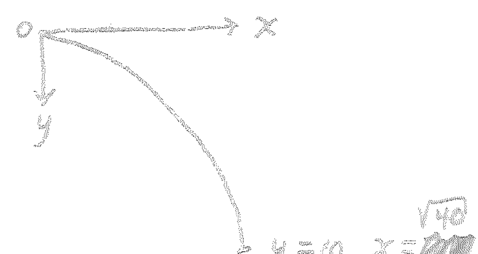
$$\Delta F = 2\pi G m M t\left(1 - \frac{\Delta h}{\sqrt{a^2 + h^2}}\right)$$

$$F = \int_0^h 2\pi G m M t\left(1 - \frac{dh'}{\sqrt{a^2+(dh)^2}}\right) dh$$

$$= 2\pi G m M t\, h \Big|_0 - \int_0^h \frac{2\pi G m M t}{\sqrt{a^2+(dh)^2}}$$

4. If a particle slides down the curve of $y = x^2/4$ and it is given a shove to start it at an initial velocity of 8 feet/second, find the time it takes to go from $y = 0$ to $y = 10$.



$$t = \int_{y=y_0}^{y=y_t} \frac{ds}{\sqrt{64(y-y_0)+v_0^2}}$$

10

$\cancel{\int}$

$$Y = \frac{x^2}{4}$$

$$Y' = \frac{x}{2}$$

$$s = \int_0 \sqrt{1 + \frac{x^2}{4}}\, dx$$

$\theta = \frac{x}{2}$

$$ds = \sqrt{1+\frac{x^2}{4}}\, dx$$

$$x = 2Y^{\frac{1}{2}}$$

$$dx = \frac{1}{\sqrt{Y}}\, dY$$

$$t = \frac{1}{24}\int_0^{10} (1+Y)^{\frac{1}{2}} Y^{-\frac{1}{2}}\, dY$$

$$t = \int_0^{10} \frac{\sqrt{1+\frac{x^2}{4}}}{\sqrt{Y}\sqrt{64(Y-Y_0)+V_0^2}}\, dY$$

$$= \int_0^{10} \frac{\sqrt{1+Y}}{\sqrt{Y}\sqrt{64(10)-64}}\, dY$$

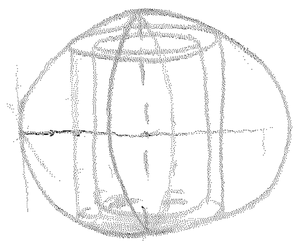
$$= \int_0^{10} \frac{(1+Y)^{\frac{1}{2}}}{24Y^{\frac{1}{2}}}\, dY =$$

$\cancel{(1+Y)^{\frac{3}{2}}}$

15F25 April 1969 CALCULUS III Name Bob Marks Box No. 156

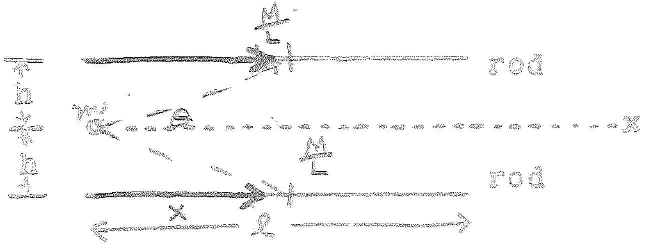
70

1. Find the volume generated by revolving the area bounded by the parabola $y^2 = 8x$ and its latus rectum about the latus rectum. Use the shell method.



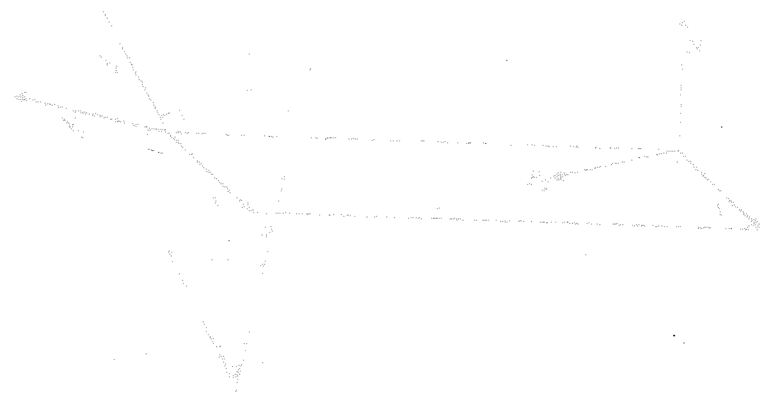
$$dV = 2\pi r\, 2y\, dx$$
$$r = (2-x)$$
$$V = \int_0^2 2\pi(2-x)(2)\,2\sqrt{2}\,x^{\frac{1}{2}}\,dx$$
$$= 8\sqrt{2}\,\pi \int_0^2 \left(2x^{\frac{1}{2}} - x^{\frac{3}{2}}\right)dx$$
$$= 8\sqrt{2}\,\pi\left[\tfrac{4}{3}x^{\frac{3}{2}} - \tfrac{2}{5}x^{\frac{5}{2}}\right]_0^2$$
$$= 8\sqrt{2}\,\pi\left[\tfrac{4}{3}\,2\sqrt{2} - \tfrac{2}{5}\,4\sqrt{2}\right]$$
$$= 8\sqrt{2}\,\pi\left(\tfrac{8\sqrt{2}}{3} - \tfrac{8\sqrt{2}}{5}\right)$$
$$= 128\pi\left(\tfrac{1}{3} - \tfrac{1}{5}\right)$$
$$= \underline{\frac{256\pi}{15}}$$

2. Find the force exerted on the mass m by the two parallel rods as shown in the figure. Each rod has a uniformly distributed mass M.



—5

$$dF = 2Gm\frac{M}{Lr^2}\cos\theta\, dx \qquad \cos\theta = \frac{h}{\sqrt{h^2+x^2}}$$
$$F = 2Gm\frac{M}{L}\int_0^{\ell}\frac{x}{\sqrt{h^2+x^2}\,(h^2+x^2)}$$
$$= 2Gm\frac{M}{\ell}\int_0^{\ell}\frac{x}{(h^2+x^2)^{3/2}}$$
$$= 2Gm\frac{M}{\ell}\,(h^2+x^2)^{-\frac{1}{2}}\Big|_{\ell}^{0} \quad \text{or} \quad -(h^2+x^2)^{-\frac{1}{2}}\Big|_0^{\ell}$$
$$= -2Gm\frac{M}{\ell}\left[(h^2+\ell^2)^{-\frac{1}{2}} - \frac{1}{h}\right]$$
$$= \frac{-2GmM\left(h-\sqrt{h^2+\ell^2}\right)}{\ell\, h\sqrt{h^2+\ell^2}}$$
$$= \frac{2GmM}{\ell}\left[\frac{1}{h} - \frac{1}{\sqrt{h^2+\ell^2}}\right]$$ is the usual way to write the ans.



Problem 4. Find an equation of the plane that passes through the two points

$$P_1(1, 0, -1) \quad \text{and} \quad P_2(-1, 2, 1)$$

and is parallel to the line of intersection of the planes $3x + y - 2z = 6$ and $4x - y + 3z = 0$.

Solution. Our main problem is to find a vector

$$\mathbf{N} = \overrightarrow{P_1P_2} \times \mathbf{V},$$

normal to the plane in question (see Fig. 12.59). The line of intersection of the two given planes is parallel to the vector

$$\mathbf{V} = \mathbf{N}_1 \times \mathbf{N}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & -2 \\ 4 & -1 & 3 \end{vmatrix} = \mathbf{i} - 17\mathbf{j} - 7\mathbf{k},$$

where $\mathbf{N}_1$ and $\mathbf{N}_2$ are normals to the two given planes. The vector $\overrightarrow{P_1P_2} = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ is to lie in the required plane. Now we can move the line $\mathbf{V}$ parallel to itself until it also lies in the required plane (since the plane is to be parallel to $\mathbf{V}$). Hence we may take

$$\mathbf{N} = \overrightarrow{P_1P_2} \times \mathbf{V} = 20\mathbf{i} - 12\mathbf{j} + 32\mathbf{k}$$

as a vector normal to the plane. Actually

$$\tfrac{1}{4}\mathbf{N} = 5\mathbf{i} - 3\mathbf{j} + 8\mathbf{k}$$

would serve as well. From this normal vector, we may conclude

$$A = 5, \quad B = -3, \quad C = 8.$$

Putting these together with $x_1 = 1$, $y_1 = 0$, $z_1 = -1$, since $P_1(1, 0, -1)$ lies in the plane, the required plane is therefore

$$5(x - 1) - 3(y - 0) + 8(z + 1) = 0$$

or

$$5x - 3y + 8z + 3 = 0.$$

EXERCISES

1. Find the coordinates of the point P in which the line

$$\frac{x - 1}{2} = \frac{y + 1}{-1} = \frac{z}{3}$$

intersects the plane $3x + 2y - z = 5$ [illegible].

2. Find parametric and Cartesian equations of the line joining the points $A(1, 2, -1)$ and $B(-1$, [illegible].

3. Show, by vector methods, that the distance from the point $P_1(x_1, y_1, z_1)$ to the plane

$$Ax + By + Cz - D = 0$$

is

$$\frac{|Ax_1 + By_1 + Cz_1 - D|}{\sqrt{A^2 + B^2 + C^2}}.$$

4. (a) Define what is meant by the angle between a line and a plane.

(b) Find the angle between the line [illegible]

and the plane $2x - 2y$ [illegible].

[illegible] planes. What is the [illegible] of the planes corresponding [illegible] and the other to [illegible]

[illegible]

[illegible] through [illegible] and perpendicular [illegible] intersection of the planes

[illegible]

[illegible] through the points

[illegible]

[illegible] distance from the origin to the line

[illegible]

[illegible] find the direction [illegible] of the line

[illegible]

19. The equation [illegible] through P_1 [illegible] inequality [illegible] for your answer.

20. The unit vector [illegible] with the positive [illegible] is normal to a [illegible] passing through [illegible]

12.5 PRODUCTS OF THREE OR MORE VECTORS

Vector products involving three or more vectors arise in physics and engineering problems. For example (see Sears, [illegible] and Zemansky, [illegible] 1960), the electromotive force [illegible] a conductor [illegible] through a magnetic field [illegible] density is $\mathbf{B}$ is given by [illegible] Here the [illegible] in parentheses is a vector, [illegible] forming the [illegible] vector [illegible] a scalar. [illegible] represent the result [illegible] removes the necessity [illegible] the sine of the angle between $\mathbf{B}$ and [illegible] the angle between [illegible] the velocity vector $\mathbf{v}$. [illegible]

7) $\frac{X-1}{2} = \frac{Z}{3} = \frac{Y+1}{-1}$

$Z = \frac{3(X-1)}{2}$

$Y = \frac{1-X}{2} - 1$

$3X + 2Y - Z = 5$

$3X + (1 - X - 2) - \frac{3(X-1)}{2} = 5$

$3X + 1 - X - 2 - \frac{3X}{2} + \frac{1}{2} = 5$

$\frac{1}{2}X = \frac{11}{2}$

$X = 11$

$\frac{X-1}{2} = \frac{Z}{3}$

$5 = \frac{Z}{3}$

$Z = 15$

$-5 = Y + 1$

$Y = -6$

$P(11, -6, 15)$

8) $A + B - C = D$

$2A + 0 + 2C = D$

$0 - 2B + C = D$

$$D = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 0 & 2 \\ 0 & -2 & 1 \end{vmatrix} = 4 + 4 - 2 = 6$$

$$D_1 = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 0 & 2 \\ 1 & -2 & 1 \end{vmatrix} = 2 + 2 + 4 - 1 = 7$$

$$D_2 = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 1 & 2 \\ 0 & 1 & 1 \end{vmatrix} = 1 - 2 - 2 - 2 = -5$$

$$D_3 = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & -2 & 1 \end{vmatrix} = -4 + 2 - 2 = -4$$

$A = \frac{D_1}{D} = \frac{7}{6}$ $\quad B = \frac{D_2}{D} = \frac{-5}{6}$ $\quad C = \frac{-4}{6} = \frac{D_3}{D}$

$\frac{7}{6}X - \frac{5}{6}Y - \frac{4}{6}Z = 1$

$7X - 5Y - 4Z = 6$

$$2X + Y - Z = 3$$
$$X + 2Y + Z = 2$$
$$3X + 3Y = 5$$
$$X = \frac{5-3Y}{3} \qquad Y = \frac{5-3X}{3} = \frac{3Z+1}{3}$$
$$\frac{5-3Y}{3} + 2Y - 2 = Z$$
$$\frac{5}{3} - \cancel{Y} + \cancel{2}Y - 2 = Z$$
$$Y = Z + \frac{1}{3}$$
$$3Y = 5 - 3X = 3Z + 1$$
$$AX + BY + CZ = 1$$
$$2A + B - C = 1$$
$$(A,B,C)_{DIR} = (1, \tfrac{2}{3}$$

11)
$$2X + Y - Z = 3$$
$$X + 2Y + Z = 2$$
$$3X + 3Y = 5$$
$$X = \frac{5-3Y}{3}$$
$$\frac{5}{3} - \cancel{Y} + \cancel{2}Y + Z = 2$$
$$Y = Z + \frac{1}{3}$$
$$= \frac{3Z+1}{3}$$
$$3Y = 3Z + 1 = 5 - 3X$$
$$(A,B,C)_{DIR} = (1, -1, -1)$$
$$A(X - X_0) + B(Y - Y_0) + C(Z - Z_0) = 0$$
$$1(X-2) - 1(Y-1) - 1(Z+1) = 0$$
$$X - 2 - Y \cancel{+1} - Z \cancel{-1} = 0$$
$$X - Y - Z = 2$$

14) a) $\overline{AC} \times \overline{AB} = \vec{0} \Leftrightarrow$

(if $\overline{AC}$ and $\overline{AB} \neq 0 \Leftrightarrow$)

$\sin\theta = 0 \Leftrightarrow$

$\theta = 0°$ or $180° \Leftrightarrow$

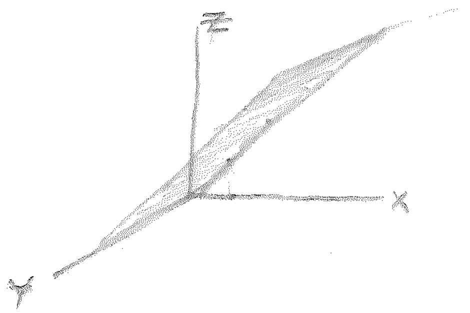
lines are colinear

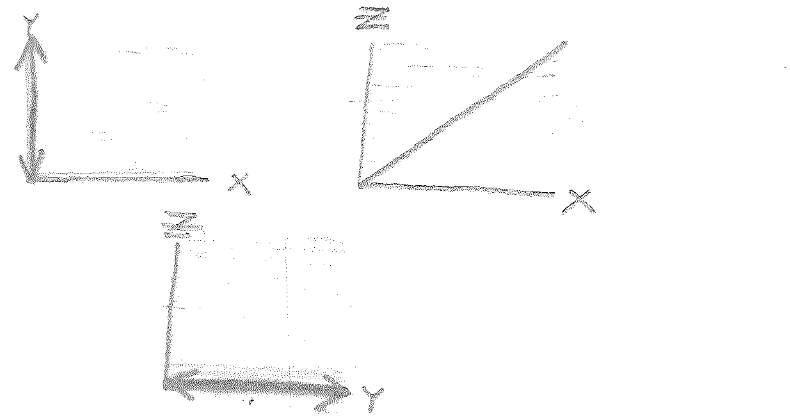
b) $D = \begin{vmatrix} 1 & 2 & -3 \\ 3 & 1 & 0 \\ -3 & 4 & -9 \end{vmatrix} = -9 - 36 + 45 - 9 \neq 0$

No

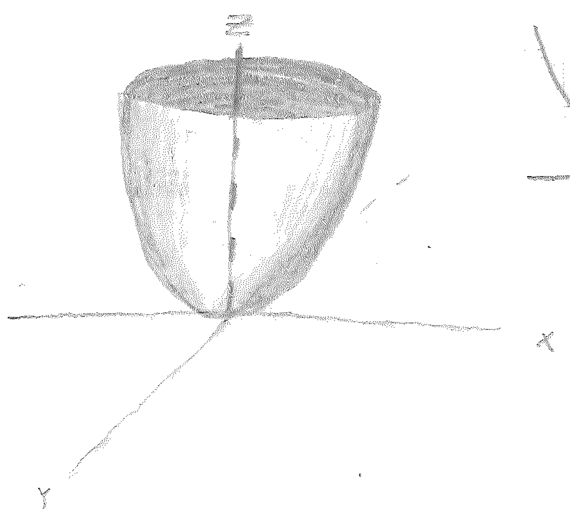
Pg 500

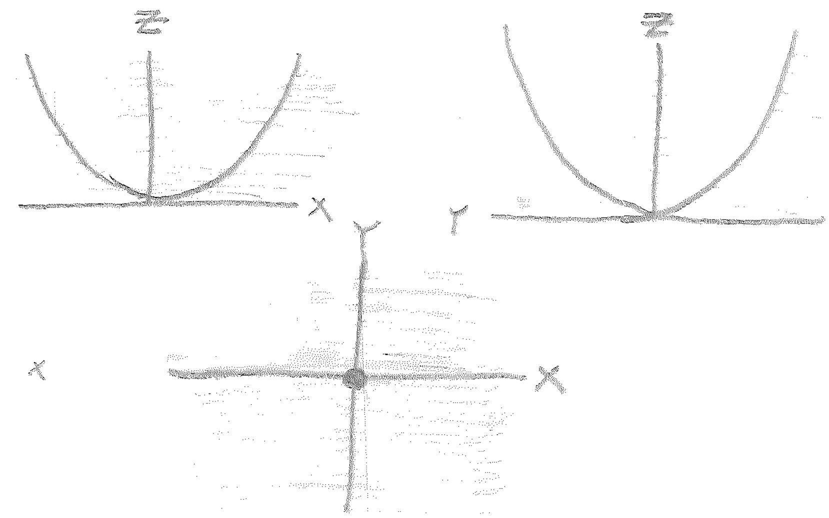
1) $f(X,Y) = Z = X$





4) $f(X,Y) = X^2 - Y^2$





6) $w = e^X \cos Y$

$\frac{dw}{dX} = (\cos Y)\, e^X$

$\frac{dw}{dY} = -e^X \sin Y$

9) $w = \ln\sqrt{X^2 + Y^2}$

$\frac{dw}{dX} = \frac{2X}{2\sqrt{X^2+Y^2}\,\sqrt{X^2+Y^2}}$

$= \frac{X}{X^2+Y^2}$

$\frac{dw}{dY} = \frac{Y}{Y^2+X^2}$

11) $f(X,Y,Z,w) = X^2 e^{2Y+3Z} \cos 4w$

$= U$

$\frac{dU}{dX} = 2X e^{2Y+3Z} \cos 4w$

$\frac{dU}{dY} = X^2\, 2 e^{2X+3Z} \cos 4w$

$\frac{dU}{dZ} = X^2\, 3 e^{2Y+3Z} \cos 4w$

$\frac{dU}{dw} = -4X^2 e^{2Y+3Z} \sin 4w$

14) $U = f(r, \theta, z) = \dfrac{r(2 - \cos 2\theta)}{r^2 + z^2}$

$$U = \frac{2r}{r^2+z^2} - \frac{\cos 2\theta}{r^2+z^2}$$

$$\frac{dU}{d\theta} = \frac{2 \sin 2\theta}{r^2+z^2}$$

$$U = \frac{r(2-\cos 2\theta)}{r^2+z^2} \qquad V = \frac{1}{r^2+z^2} \qquad U = \frac{r(2-\cos 2\theta)}{V}$$

$$\frac{dV}{dz} = \frac{-2z}{(r^2+z^2)^2} \qquad \frac{dU}{dV} = \frac{-r(2-\cos 2\theta)}{V^2}$$

$$\frac{dU}{dz} = \frac{-2z}{(r^2+z^2)^2} \cdot (-r)(2-\cos 2\theta)(r^2+z^2)^2$$

$$= 2zr(2-\cos 2\theta)$$

$$U = \frac{r(2-\cos 2\theta)}{r^2+z^2}$$

$$\ln U = \ln r(2-\cos 2\theta) - \ln(r^2+z^2)$$

$$= \ln r + \ln(2-\cos 2\theta) - \ln(r^2+z^2)$$

$$\frac{\dot{U}}{U} = \frac{1}{r} - \frac{2r}{r^2+z^2} = \frac{r^2+z^2-2r^2}{r(r^2+z^2)} = \frac{z^2-r^2}{r(r^2+z^2)}$$

$$\frac{dU}{dr} = \frac{(z^2-r^2)}{\cancel{r}(r^2+z^2)} \cdot \frac{\cancel{r}(2-\cos 2\theta)}{(r^2+z^2)}$$

$$= \frac{(z^2-r^2)(2-\cos 2\theta)}{(r^2+z^2)^2}$$

15) $f(X, Y, U, V) = Z = \dfrac{X^2+Y^2}{U^2+V^2}$

$$Z = \frac{X^2}{U^2+V^2} + \frac{Y^2}{U^2+V^2}$$

$$\frac{dZ}{dX} = \frac{2X}{U^2+V^2} \qquad \frac{dZ}{dY} = \frac{2Y}{U^2+V^2}$$

$$Z = \frac{X^2+Y^2}{U^2+V^2}$$

$$\frac{dZ}{dU} = \frac{-2U(X^2+Y^2)}{(U^2+V^2)^2} \qquad \frac{dZ}{dV} = \frac{-2V(X^2+Y^2)}{(U^2+V^2)^2}$$

PROVE

$$\cosh^2 X + \sinh^2 X = \cosh 2X$$

PROVE

$$\cosh 2X = \tfrac{1}{2}(e^{-2X} + e^{2X})$$

$$\cosh 2X = \cosh^2 X + \sinh^2 X$$

$$= \frac{(e^{x}+e^{-x})^2 + (e^{x}-e^{-x})^2}{4}$$

$$= \frac{e^{2X}+2+e^{-2X}+e^{2X}-2+e^{-2X}}{4}$$

$$= \frac{2e^{2X}+2e^{-2X}}{4}$$

$$= \frac{e^{2X}+e^{-2X}}{4}$$

PROVE

$$\sinh 2X = 2\sinh X \cosh X = \frac{e^{2X}-e^{-2X}}{2}$$

$$\sinh 2X = 2\left(\frac{e^{x}-e^{-x}}{2}\right)\left(\frac{e^{x}+e^{-x}}{2}\right)$$

$$= \frac{e^{2X}-e^{-2X}}{2}$$

Pg 500

10) $w = \cosh \frac{Y}{X}$

$$\frac{\partial w}{\partial Y} = \frac{+1}{X} \sinh \frac{Y}{X}$$

$$\frac{\partial w}{\partial X} = \left(\frac{+Y}{X^2}\right) \sinh \frac{Y}{X}$$

16) $z = f(X, Y, r, s) = \sin 2X \cosh 3r + \sinh 3Y \cos 4s$

$$\frac{\partial z}{\partial X} = 2\cos 2X \cosh 3r$$

$$\frac{\partial z}{\partial Y} = 3\cosh 3Y \cos 4s$$

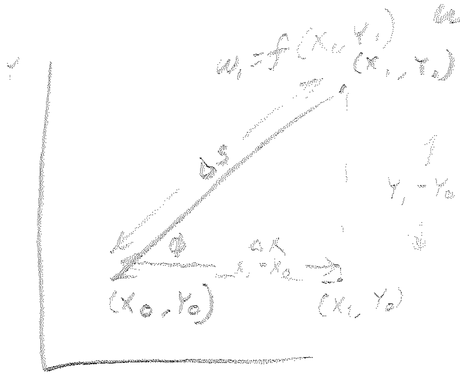
$$\frac{\partial z}{\partial r} = 3\sin 2X \sinh 3r$$

$$\frac{\partial z}{\partial s} = 4\sinh 3Y \cos 4s$$

DIRECTIONAL DERIVITIVE

502 : 1, 3, 5
502 : 2, 6
510 ; 1, 2, 4

$w = f(x, y)$



$\frac{dw}{dx} = \lim \frac{f(x_1, \ldots) - f(x_0, y_0)}{x_1 - x_0}$

$$\frac{dw}{ds} = \lim_{\Delta s \to 0} \frac{w_1 - w_0}{\Delta s} = \frac{d\left(f(x_1, y_1) - f(x_0, y_0)\right)}{dx}$$

$$= \frac{f(x_1, y_1) - f(x_1, y_0) + f(x_1, y_0) - f(x_0, y_0)}{\Delta s}$$

$f(x_0 + \Delta x, y_0 + \Delta y) - f \ldots$

$$= \frac{f(x_1, y_0) - f(x_0, y_0)}{x_1 - x_0} \quad \frac{\Delta x}{\Delta s}^{= \cos\phi} +$$

$$\frac{f(x_1, y_1) - f(x_1, y_0)}{y_1 - y_0} \quad \frac{\Delta y}{\Delta x} = \sin\phi$$

Mean Value (for cont. functions)

$$f(x_1) - f(x_0) = (x_1 - x_0) f'(\bar{x})$$
$$x_1 < \bar{x} < x_0$$

$$= \frac{\partial f}{\partial x}(\bar{x}, y_0) \cos\phi + \frac{\partial f}{\partial x}(x_1, \bar{y}) \sin\phi$$

DON'T HOLD FOR DISC.

$$\therefore \frac{dw}{ds} = \cos\phi \left(\frac{\partial f}{\partial x}\right)_0 + \sin\phi \left(\frac{\partial f}{\partial y}\right)_0$$

STEEPEST PATH

$W = 100 - x^2 - y^2$

$P(3, 4)$

WHAT IS $\left(\frac{dw}{ds}\right)_{MAX}$

$\frac{\partial w}{\partial x} = -2x$

$\frac{dw}{ds} = \cos\phi(-6) + \sin\phi(-8)$

IF W = K

$g(\phi) = \frac{dw}{ds}$

$g'(\phi) = 6\sin\phi - 8\cos\phi \Rightarrow$

$\tan\phi = \frac{4}{3}$

$\sin\phi = \frac{4}{5} \quad \cos\phi = \frac{3}{5}$

502 56
572 :35
4,12,15

5) $z = \frac{x}{\sqrt{x^2+y^2}}$

$\frac{\partial z}{\partial x} = \frac{y^2}{(x^2+y^2)^{\frac{3}{2}}}$

$\frac{\partial z}{\partial y} = \frac{-xy}{(x^2+y^2)^{\frac{3}{2}}}$

EQ OF PLAIN

$$(x-x_0)\frac{\partial z}{\partial x}\Big|_0 + (y-y_0)\frac{\partial z}{\partial y}\Big|_0 - (z-z_0) = 0$$

PLUG & CHUG

$$w = f(x, y, z)$$

$$\frac{dw}{ds} = \lim_{\Delta s \to 0} \frac{w_1 - w_0}{\Delta s}$$

$$\frac{f(x_0+\Delta x,\ y_0+\Delta y,\ z_0+\Delta z) - f(x_0, y_0, z_0)}{\Delta s} =$$

$$\frac{f(x_0+\Delta x,\ y_0+\Delta y,\ z_0+\Delta z) - f(x_0,\ y_0+\Delta y,\ z_0+\Delta z)}{\Delta x}\frac{\Delta x}{\Delta s} +$$

$$\frac{f(x_0,\ y_0+\Delta y,\ z_0+\Delta z) - f(x_0, y_0, z_0+\Delta z)}{\Delta y}\frac{\Delta y}{\Delta s} +$$

$$\frac{f(x_0, y_0, z_0+\Delta z) - f(x_0, y_0, z_0)}{\Delta z}\frac{\Delta z}{\Delta s}$$

$$\frac{\Delta z}{\Delta s} = \cos\gamma$$

$$\frac{df}{ds}\Big|_0 = \frac{\partial f}{\partial x}\Big|_0 \cos\alpha + \frac{\partial f}{\partial x}\Big|_0 \cos\beta + \frac{\partial f}{\partial z}\Big|_0 \cos\gamma$$

$$\tilde{\imath}\cdot\nabla w = \frac{dw}{ds} = \frac{\partial w}{\partial x}\cos\alpha + \frac{\partial w}{\partial y}\cos\beta + \frac{\partial w}{\partial z}\cos\gamma$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

equapotential surface w is const of $w = f(x, y, z)$
Δw at P is $\perp$ ~~to~~ equapot. surface
at pt P

15-3
Pg 502

1) $z = x^2 + y^2 \quad (3,4,25)$

$\frac{\partial z}{\partial x} = 2x = 6 \qquad \frac{\partial z}{\partial y} = 2y = 8$

$6(x-3) + 8(y-4) = z - 25$

$\underline{6x + 8y = z + 25}$

$\frac{x-3}{6} = \frac{y-4}{8} = \frac{z-25}{-1}$

3) $z = x^2 - xy - y^2 \quad (1,1,-1)$

$\frac{\partial z}{\partial x} = 2x - y = 1 \qquad \frac{\partial z}{\partial y} = -x - 2y = -3$

$(x-1) - 3(y-1) = z + 1$

$x - 1 - 3y + 3 = z + 1$

$\underline{x - 3y = z - 1}$

$\frac{x-1}{1} = \frac{y-1}{-3} = \frac{z+1}{-1}$

6) $x = e^{2y-z} \quad (1,1,2)$

$\frac{\partial x}{\partial y} = 2e^{2y-z} = 2 \qquad \frac{\partial x}{\partial z} = -e^{2x-z} = -1$

$2(y-1) - (z-2) = x - 1$

$2y - 2 - z + 2 = x - 1$

$2y - z = x - 1$

$\frac{y-1}{2} = \frac{z-2}{-1} = \frac{x-1}{-1}$

$z^2 = 2x^2 + 4y^2$

$z = (2x^2 + 4y^2)^{\frac{1}{2}}$

$\frac{\partial z}{\partial x} = \frac{2x}{(2x^2+4y^2)^{\frac{1}{2}}}$

$\frac{\partial z}{\partial y} = \frac{4y}{(2x^2+4y^2)^{\frac{1}{2}}}$

$\frac{2x_0}{(2x_0^2+4y_0^2)^{\frac{1}{2}}}(x - x_0) = \frac{4y_0\,(y-y_0)}{(2x_0^2+4y_0^2)^{\frac{1}{2}}} = \frac{z - z_0}{-1}$

15-5
Pg 510

1) $f = e^x \cos yz \quad (0,0,0)$

$A = 2i + j - 2k$

$\frac{\partial f}{\partial x} = e^x \cos yz = 1$

$\frac{\partial f}{\partial y} = -z e^x \sin yz = 0$

$\frac{\partial f}{\partial z} = -y e^x \sin yz = 0$

$\frac{2i + j - 2k}{|A| = 3}$

2

$\frac{2}{3}$

3) $f = x^2 + 2y^2 + 3z^2 \quad (1,1,1)$

$\frac{\partial f}{\partial x} = 2x = 2$

$\frac{\partial f}{\partial y} = 4y = 4 \qquad \frac{\partial f}{\partial z} = 6z = 6$

$2i + 4j + 6k$

$\frac{i + j + k}{2 + 4 + 6 = 12}$

$|A| = \sqrt{3}$

$\frac{12}{\sqrt{3}} = 4\sqrt{3}$

5) $f = (x+y-2)^2 + (3x-y-6)^2$

$(1,1,0)$

$\frac{\partial f}{\partial x} = 2(x+y-2) + 6(3x-y-6) = -24$

$\frac{\partial f}{\partial y} = 2(x+y-2) - 2(3x-y+6) = 8$

$-24i + 8j$

5.7 pg 517

1) $w = x^2 + y^2 + z^2$

$x = e^t \cos t \quad y = e^t \sin t \quad z = e^t$

$\frac{\partial w}{\partial x} = 2x \quad \frac{\partial x}{\partial t} = e^t(\cos t - \sin t) \qquad \frac{\partial w}{\partial t} = 2x\, e^t(\cos t - \sin t)$

$\frac{\partial w}{\partial y} = 2y \quad \frac{dy}{dt} = e^t(\sin t + \cos t) \qquad \frac{\partial w}{\partial t} = 2y\, e^t(\sin t + \cos t)$

$\frac{\partial w}{\partial z} = 2z \quad \frac{\partial z}{\partial t} = e^t \qquad \frac{\partial w}{\partial t} = 2z e^t$

$\frac{dw}{dt} = 2e^t\left[x(\cos t - \sin t) + y(\sin t + \cos t) + z\right]$

3) $w = e^{2x+3y} \cos 4z$

$\frac{\partial w}{\partial x} = 2e^{2x+3y} \cos 4z \quad \frac{\partial x}{\partial t} = \frac{1}{t}$

$\frac{\partial w}{\partial y} = 3e^{2x+3y} \cos 4z \quad \frac{\partial y}{\partial t} = \frac{t}{t^2+1}$

$\frac{\partial w}{\partial z} = -4e^{2x+3y} \sin 4z \quad \frac{\partial y}{\partial z} = 1$

$\frac{dw}{dz} = e^{2x+3x}\left[\left(\frac{2}{t}\cos 4z\right) + \frac{3t}{t^2+1}\cos 4z - 4\sin 4z\right]$

5) $\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$

$\frac{\partial w}{\partial x} = \frac{x}{(x^2+y^2+z^2)^{\frac{1}{2}}} \quad \frac{\partial x}{\partial r} = e^r \cos s$

$\frac{\partial w}{\partial y} = \frac{y}{(x^2+y^2+z^2)^{\frac{1}{2}}} \quad \frac{\partial y}{\partial r} = e^r \sin s$

$\frac{\partial w}{\partial z} = \frac{z}{(x^2+y^2+z^2)^{\frac{1}{2}}} \quad \frac{\partial z}{\partial r} = 1$

$\frac{\partial w}{\partial r} = \frac{1}{x^2+y^2+z^2}\left[x e^r \cos s + y e^r \sin s + z\right]$

EX 15.9 Pg 523-4

1) $Z = X^2 + XY + Y^2 + 3X - 3Y + 4$

$\frac{\partial Z}{\partial X} = 2X + Y + 3$

$\frac{\partial Z}{\partial Y} = X + 2Y - 3$

$\frac{\partial^2 Z}{\partial X^2} = 2 \quad \frac{\partial^2 Z}{\partial Y} = 2$

MINIMUM AT $(^-3, 3, ^-5)$

$D = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3 \qquad X = ^-3$

$D_X = \begin{vmatrix} ^-3 & 1 \\ 3 & 2 \end{vmatrix} = ^-9$

$D_Y = \begin{vmatrix} 2 & ^-3 \\ 1 & 3 \end{vmatrix} = 9 \qquad Y = 3$

4) $Z = 2XY - 5X^2 - 2Y^2 + 4X + 4Y - 4$

$\frac{\partial Z}{\partial X} = 2Y - 5X + 4 = 0$

$\frac{\partial Z}{\partial Y} = 2X - 4Y + 4 = 0$

$\frac{\partial^2 Z}{\partial X^2} = ^-5 \quad \frac{\partial^2 Z}{\partial Y^2} = ^-4$

MAX AT $(^-2, 0, ^-32)$

$D = \begin{vmatrix} 2 & ^-5 \\ 2 & ^-4 \end{vmatrix} = 2$

$D_X = \begin{vmatrix} ^-4 & ^-5 \\ ^-4 & ^-4 \end{vmatrix} = ^-4$

$D_Y = \begin{vmatrix} 2 & ^-4 \\ 2 & ^-4 \end{vmatrix} = 0$

6) $Z = Y^2 + XY - 2X - 2Y + 2$

$\frac{\partial Z}{\partial X} = Y - 2 = 0 = \frac{\partial Z}{\partial Y} = 2Y + X - 2$

$\frac{\partial^2 Z}{\partial X^2} = 0 \quad \frac{\partial^2 Z}{\partial Y^2} = 2$

$(2, 2, 2)$ SADDLE

$D = \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = ^-1 \qquad X = 2$

$D_X = \begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} = 2 \qquad Y = ^-2$

$D_Y = \begin{vmatrix} 0 & 2 \\ 1 & 2 \end{vmatrix} = ^-2$

15-13
7, 3, 5,
15.14
2, 5, 10

Vector problems

A Given the vectors $\vec{a} = 2\vec{i} + 4\vec{j} - 5\vec{k}$, $\vec{b} = \vec{i} + 2\vec{j} + 3\vec{k}$, ~~find~~ and $\vec{c} = 3\vec{i} + \vec{j} + 2\vec{k}$

1 find the sum $\vec{a} + \vec{b}$

2 find the difference $\vec{a} - \vec{b}$

3 obtain the magnitude $|\vec{a} + \vec{b}|$ of $\vec{a} + \vec{b}$

4 obtain a unit vector in the direction of $\vec{a} + \vec{b}$. How many such vectors do you find?

5 evaluate $\vec{a} \cdot \vec{b}$

6 evaluate $\vec{a} \times \vec{b}$

7 Show that $\vec{a}$ is perpendicular to $\vec{c}$

8 evaluate $\vec{a} \times (\vec{b} \times \vec{c})$ in two ways

9 obtain a unit vector perpendicular to both $\vec{b}$ and $\vec{c}$ (how many solutions?)

10 evaluate $\vec{a} \cdot \vec{b} \times \vec{c}$

B Prove the following

1 $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$

2 $\vec{a} \times \vec{b} \cdot \vec{c} \times \vec{d} = \vec{a} \cdot \vec{c}\, \vec{b} \cdot \vec{d} - \vec{a} \cdot \vec{d}\, \vec{b} \cdot \vec{c}$

3 $(p\vec{a} + q\vec{b}) \times (r\vec{a} + s\vec{b}) = (ps - qr)\, \vec{a} \times \vec{b}$

~~15.8~~ 15.3 SEC 15.8 4, 5, 10
15.5
15.6 Test, up to max & min of ind variables
15.7
15.8
TO 15.9

BOO MARKS

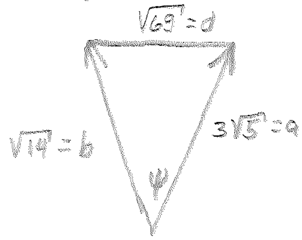
F

156

1) $\vec{a}+\vec{b} = 3\vec{i} + 6\vec{j} - 2\vec{k}$
2) $\vec{a}-\vec{b} = \vec{i} + 2\vec{j} - 8\vec{k}$
3) $|a+b| = \sqrt{9+36+4}$
$= 7$

4) ?
5) $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\psi = a_1b_1 + a_2b_2 + a_3b_3$ P!

$|\vec{a}-\vec{b}| = d = \sqrt{69}$

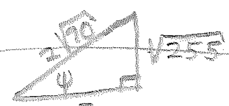


$\cos\psi = \dfrac{d^2-b^2-a^2}{2ba}$

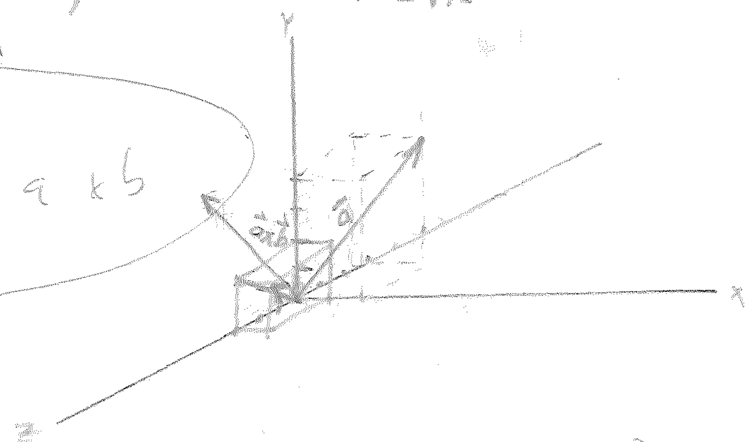
$= \dfrac{69-14-45}{6\sqrt{70}} = \dfrac{10}{6\sqrt{70}} = \dfrac{5}{2\sqrt{70}}$

$\vec{a}\cdot\vec{b} = (3\sqrt{5})(\sqrt{14})\dfrac{5}{2\sqrt{70}} = \dfrac{15}{2}$

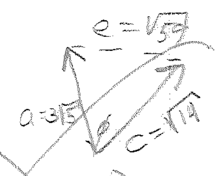
6) $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\psi = (3\sqrt{5})(\sqrt{14})\dfrac{\sqrt{255}}{2\sqrt{70}} = \dfrac{3\sqrt{255}}{2}$



$\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = a \times b$



7) $|\vec{a}| = 3\sqrt{5}$
$|\vec{c}| = \sqrt{14}$
$\vec{a}-\vec{c} = -\vec{i} + 3\vec{j} - 7\vec{k}$
$e = \sqrt{59}$



$a^2 + c^2 = e^2 \Rightarrow a \perp c$

$(\vec{b}\times\vec{c})$

8) $\vec{a} \times (\vec{b} \times \vec{c})$

$\vec{b} \times \vec{c}$

$|\vec{b}| = \sqrt{14}$

$|\vec{c}| = \sqrt{14}$

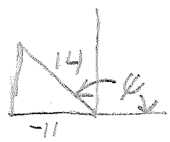
$\vec{b} - \vec{c} = -2\vec{i} + \vec{j} + \vec{k}$

$f = \sqrt{6}$

$$\cos \varphi = \frac{f^2 - b^2 - c^2}{2bc}$$

$$= \frac{6 - 28}{28} = \frac{-22}{28} = \frac{-11}{14}$$

$\sqrt{184}$ -



$\vec{b} \times \vec{c} = 14 \ \frac{\sqrt{184}}{14}$

$= 2\sqrt{46}$

$$|b \times c| = (\vec{i} + 2\vec{j} + 3\vec{k}) \times (3\hat{i} + \hat{j} + 2\hat{k})$$

$$= \cancel{3\hat{i}\hat{i}} + 3\hat{i}\times\hat{j} + 2\hat{k}\times j$$

$$+ 2\hat{j}\times 3\hat{i} + \cancel{2\hat{j}\times\hat{j}} + 2\hat{j}\times 2\hat{k}$$

$$+ 3\hat{k}\times 3\hat{i} + 3\hat{k}\times j + \cancel{3\hat{k}\times 2\hat{k}}$$

$= (3 + 2 + 6 + 9 + 9 + 3) = 27$

$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{b})\vec{c} + (\vec{a} \cdot \vec{c})\vec{b}$

$= \frac{15}{2}\sqrt{111} + 70\sqrt{3}$

$|\vec{a}| = 5\sqrt{3}$

$|\vec{c}| = |\vec{b}| = \sqrt{14}$

BOB MARKS

PROOFS

1) $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$

ASSUME

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$$

$$\vec{a} \times (|\vec{b}||\vec{c}| \sin\psi) = (|\vec{a}||\vec{b}| \sin\phi) \times c$$

$$|\vec{a}|\,\big||\vec{b}||\vec{c}| \sin\psi\big| \sin\theta = |\vec{a}|\,\big||\vec{b}||\vec{c}| \sin\phi\big| \sin\gamma$$

2) $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c})$

$$(|\vec{a}||\vec{b}| \sin\phi) \cdot (|\vec{c}||\vec{d}| \sin\psi) = (|\vec{a}||\vec{c}| \cos\theta)(|\vec{b}||\vec{d}| \cos\gamma) - (|\vec{a}||\vec{d}| \cos\beta)(|\vec{b}||\vec{c}| \cos\varepsilon)$$

don't use component form

[illegible] Ratio Tests

<u>Theorem</u> A. If $\sum_{k=1}^{\infty} v_k$ is a convergent positive series and if the terms of another series $\sum_{k=1}^{\infty} u_k$ have the property that $\frac{u_{k+1}}{u_k} \le \frac{v_{k+1}}{v_k}$, then $\sum_{k=1}^{\infty} u_k$ is also convergent.

B. If $\sum_{k=1}^{\infty} v_k$ is a divergent positive series and if the terms of another series $\sum_{k=1}^{\infty} u_k$ have the property that $\frac{u_{k+1}}{u_k} \ge \frac{v_{k+1}}{v_k}$, then $\sum_{k=1}^{\infty} u_k$ is also convergent.

Proof A. We know that $\frac{u_2}{u_1} < \frac{v_2}{v_1}$, $\frac{u_3}{u_2} < \frac{v_3}{v_2}$ ... $\frac{u_k}{u_{k-1}} < \frac{v_k}{v_{k-1}}$

Multiply all inequalities to obtain $\frac{u_k}{u_1} < \frac{v_k}{v_1}$ or $u_k < \frac{u_1}{v_1} v_k$

According to exercise 1, p 632 $\sum (c v_k)$ converges iff $\sum v_k$ converges. The terms of $\sum u_k$ hence are less than the terms of a convergent series. It then follows from the comparison theorem that $\sum u_k$ must be convergent.

Proof B. We know that $\frac{u_2}{u_1} > \frac{v_2}{v_1}$, $\frac{u_3}{u_2} > \frac{v_3}{v_2}$ ... $\frac{u_k}{u_{k-1}} > \frac{v_k}{v_{k-1}}$

Multiply these inequalities to obtain $\frac{u_k}{u_1} > \frac{v_k}{v_1}$ or $u_k > \frac{u_1}{v_1} v_k$

According to exercise 1 p 632 $\sum (c v_k)$ diverges iff $\sum v_k$ diverges. The terms of $\sum u_k$ hence are greater than the corresponding terms of a divergent series. It then follows from the comparison theorem that $\sum u_k$ must be divergent.

<u>Corollary</u>. If $v_k = ar^{k-1}$, then $\sum v_k$ is the geometric series which converges for $r < 1$ and diverges for $r > 1$. Hence $\sum u_k$ converges if $\frac{u_{k+1}}{u_k} \le r < 1$ and diverges if $\frac{u_{k+1}}{u_k} \ge r > 1$

D.E. I First test 10/22/63

C

Name BOB MARKS Box 156

1 Prove that $\sinh x \cosh y + \cosh x \sinh y = \sinh(x+y)$

$$\sinh x \cosh y + \cosh x \sinh y = \frac{(e^{x}-e^{-x})(e^{y}+e^{-y})+(e^{x}+e^{-x})(e^{y}-e^{-y})}{4}$$

$$= \frac{e^{x+y}+e^{x-y}-e^{y-x}-e^{-x-y}+e^{x+y}-e^{x-y}+e^{y-x}-e^{-x-y}}{4}$$

$$= \frac{e^{x+y}-e^{-(x+y)}}{2}$$

$$= \sinh(x+y)$$

8

2 Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ if

(i) $f(x,y) = e^{2x}\cos y$ $\quad \frac{\partial f}{\partial x} = 2e^{2x}\cos y$

$\frac{\partial f}{\partial y} = -e^{2x}\sin y$

(ii) $f(x,y) = \sin(xy)$ $\quad \frac{\partial f}{\partial x} = y\cos(xy)$

6

$\frac{\partial f}{\partial y} = x\cos(xy)$

(iii) $f(x,y) = \frac{x^2}{x^2+y^2}$ $\quad \frac{\partial f}{\partial x} = 2x/(x^2+y^2) - \frac{2x^3}{(x^2+y^2)^2}$

$\frac{\partial f}{\partial y} = \frac{2yx^2}{(x^2+y^2)^2}$

(iv) Find $\frac{\partial^2 f}{\partial x \partial y}$ if $f(x,y) = e^{z}\sin x\cos y$

$$\frac{\partial f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$\frac{\partial f}{\partial x} = e^{z}\cos x\cos y$

$\frac{\partial f}{\partial y} = -e^{z}\sin x\sin y$

$\frac{\partial^2 f}{\partial x \partial y} = -e^{2z}\cos x\sin x\cos y\sin y$

3 Two lines are given by $L_1: \vec{r} = \vec{a} + t\vec{p}$, $L_2: \vec{r} = \vec{b} + t\vec{q}$. State the conditions that (i) $L_1 \parallel L_2$, (ii) $L_1 \perp L_2$, (iii) $L_1 \equiv L_2$. Determine whether the lines $L_1: \vec{r} = \vec{j} + \vec{k} + t(\vec{i} + 2\vec{j} + 3\vec{k})$ and $L_2: \vec{r} = 3\vec{i} + \vec{j} + \vec{k} + t(3\vec{j} - 2\vec{k})$ are parallel, perpendicular, identical, or neither.

7. $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-2}{4}$

$x - 2y + z = 6$

$x = 6 + 2y - z$

$x = \frac{2y}{3} + \frac{5}{3}$

$6 + 2y - z = \frac{2y+5}{3}$

$18 + 6y - 3z = 2y + 5$

$0x + 4y - 3z = -13$

$-3x + 2y - 0z = 8$

$2x \quad - z = 1$

$$D = \begin{vmatrix} 0 & 4 & 3 \\ -3 & 2 & 0 \\ 2 & 0 & -1 \end{vmatrix}$$

$D = 0 + 0 + 0 - 0 + 12 - 12 = 0$

$x = \frac{D_1}{D}$, but $\frac{A}{0}$ is undefined, thus the simultaneous solving of the line yields no answer, ie they don't ~~intersect~~, & are thus // OK

$y_1 = \frac{D_2}{D}$

$z = \frac{D_3}{D}$

$z = 6 - x + 2y$

$6 - x + 2y = \frac{4y}{3} + \frac{10}{3}$

$18 - 3x + 2y = \cancel{4y} + \cancel{10}$

$8 - 3x + 2y = 0$

$y = \frac{6 - x - z}{-2} = \frac{3x - 5}{2}$

$\cancel{6 - x - z = 5 - x}$

$\cancel{-z = 1}$

$\cancel{z = 1}$

$6 - x - z = 5 - 3x$

$2x - z = -1$

shorter : normal $\vec{n}$ to plane $x - 2y + z = 6$ is $\vec{n} = \vec{i} - 2\vec{j} + \vec{k}$

direction of line $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-2}{4}$ is $\vec{b} = 2\vec{i} + 3\vec{j} + 4\vec{k}$

Since $\vec{n} \cdot \vec{b} = 0$ line perpendicular to normal, that is parallel to plane

DE I First Test ... Solutions

1. $\sinh x \cosh y + \cosh x \sinh y = \frac{e^{x}-e^{-x}}{2}\frac{e^{y}+e^{-y}}{2} + \frac{e^{x}+e^{-x}}{2}\frac{e^{y}-e^{-y}}{2} = \frac{1}{4}(e^{x+y}+e^{x-y}-e^{-x+y}-e^{-x-y})$
$+ \frac{1}{4}(e^{x+y}-e^{x-y}+e^{-x+y}-e^{-x-y}) = \frac{1}{2}(e^{x+y}-e^{-x-y}) = \sinh(x+y)$

2 (i) $\frac{\partial f}{\partial x} = 2e^{2x}\cos y$, $\frac{\partial f}{\partial y} = -e^{2x}\sin y$ (ii) $\frac{\partial f}{\partial x} = y\cos(xy)$, $\frac{\partial f}{\partial y} = x\cos(xy)$

(iii) $\frac{\partial f}{\partial x} = \frac{2x}{x^2+y^2} - \frac{2x^3}{(x^2+y^2)^2} = \frac{2xy^2}{(x^2+y^2)^2}$ $\frac{\partial f}{\partial y} = -\frac{2x^2y}{(x^2+y^2)^2}$

iv $\frac{\partial f}{\partial y} = -e^{-x}\sin x \sin y$, $\frac{\partial}{\partial x}\frac{\partial f}{\partial y} = -e^{-x}\cos x \sin y$

3 (i) $\vec{p} \parallel \vec{q}$ or $\vec{p}\times\vec{q} = 0$ (ii) $\vec{p}\cdot\vec{q} = 0$

(iii) Lines must be parallel : $\vec{p}\times\vec{q} = 0$ or $\vec{p}\parallel\vec{q}$. Lines coincide if $\vec{b}$ is on L_1 that is if $\vec{a}-\vec{b}$ is along L_1. Hence $\vec{a}-\vec{b}\parallel\vec{p}$ or $(\vec{a}-\vec{b})\times\vec{p} = 0$

(iv) $(\vec{i}+2\vec{j}+3\vec{k})\cdot(3\vec{j}-2\vec{k}) = 0$ Hence $L_1 \perp L_2$

4 (i) $(\vec{a}-\vec{b})\times(\vec{a}+\vec{b}) = \vec{a}\times\vec{a}+\vec{a}\times\vec{b}-\vec{b}\times\vec{a}-\vec{b}\times\vec{b} = \vec{a}\times\vec{b}-\vec{b}\times\vec{a} = 2\vec{a}\times\vec{b}$

(ii) $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = \vec{a}\cdot\vec{a}-\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}-\vec{b}\cdot\vec{b} = |\vec{a}|^2-|\vec{b}|^2 = 0$

(iii) $\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b}) =$
$-\vec{a}\cdot\vec{b}\,\vec{c}+\vec{a}\cdot\vec{c}\,\vec{b}-\vec{b}\cdot\vec{c}\,\vec{a}+\vec{b}\cdot\vec{a}\,\vec{c}-\vec{c}\cdot\vec{a}\,\vec{b}+\vec{c}\cdot\vec{b}\,\vec{a} = (\vec{c}\cdot\vec{b}-\vec{b}\cdot\vec{c})\vec{a}+(\vec{a}\cdot\vec{c}-\vec{c}\cdot\vec{a})\vec{b}+(\vec{b}\cdot\vec{a}-\vec{a}\cdot\vec{b})\vec{c}$
$= 0$

5 (i) $\vec{r} = 2\vec{i}+3\vec{j}+2\vec{k}+t(2\vec{i}+2\vec{j}+\vec{k})$ since $2\vec{i}+2\vec{j}+\vec{k}$ is normal to Π and hence in the direction of L.

(ii) The positionvector $\vec{r}_0$ of the point of intersection must satisfy
$(2\vec{i}+2\vec{j}+\vec{k})\cdot\vec{r}_0 = 3$ (in Π) and $\vec{r}_0 = 2\vec{i}+3\vec{j}+2\vec{k}+t(2\vec{i}+2\vec{j}+\vec{k})$.
Substitute $\vec{r}_0$: $(2\vec{i}+2\vec{j}+\vec{k})\cdot[2\vec{i}+3\vec{j}+2\vec{k}+t(2\vec{i}+2\vec{j}+\vec{k})] = 12+9t = 3 \Rightarrow t = -1$
$\vec{r}_0 = 2\vec{i}+3\vec{j}+2\vec{k}-(2\vec{i}+2\vec{j}+\vec{k}) = \vec{j}+\vec{k}$

6 If Π is parallel to L_1 and L_2, the normal to Π must be perpendicular to L_1 and L_2. A suitable normal is hence $\vec{p}\times\vec{q}$, and the equation for Π becomes $(\vec{p}\times\vec{q})\cdot(\vec{r}-\vec{c}) = 0$

4. (i) Simplify $(\vec{a}-\vec{b}) \times (\vec{a}+\vec{b})$

ii ~~$(\vec{a}-\vec{b})|\vec{a}+\vec{b}| \sin\theta$~~

(ii) If $|\vec{a}| = |\vec{b}|$ show that $(\vec{a}+\vec{b}) \cdot (\vec{a}-\vec{b}) = 0$

The directions of $(\vec{a}+\vec{b})$ would be $\perp$ to $\vec{a}-\vec{b}$ (DIAGONALS OF A RHOMBUS ARE $\perp$) Thus the cosine between the two would be 0, equating the entire equation to 0

(iii) Show that $\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = 0$

correct but not exactly as I intended.

5 Given the point $P(2,3,2)$ and the plane Π : $2x+2y+z=3$

(i) Find the equation of the line L through P and perpendicular to Π

(ii) Find the coordinates of the point of intersection of L and Π

Bonus problem. Two lines are given by $L_1 : \vec{r} = \vec{a} + t\vec{p}$ and $L_2 : \vec{r} = \vec{b} + t\vec{q}$. Find the equation of the plane Π that is parallel to both L_1 and L_2 and that passes through a point P with positionvector $\vec{c}$

16). $X + 2Y - 2Z = 5$

$5X - 2Y - Z = 0$

$6X - 3Z = 5$

$X = \frac{5+3Z}{6} \quad Z = \frac{5-6X}{-3}$

$\frac{5+3Z-12Z}{6} + 2Y = 5$

$\frac{5-9Z}{6} + 2Y = 5$

$\frac{5}{6} - \frac{3}{2}Z + 2Y = \frac{30}{6}$

$2Y = \frac{25}{6} + \frac{3}{2}Z$

$\frac{4}{3}Y - \frac{50}{18} = Z = \frac{5-6X}{-3}$

$4Y - \frac{25}{9} = 3Z = 6X - 5$

$(A_0 B_0 C_0)_{DIR} = (6, 9, 12)$

$\frac{X+3}{2} = \frac{Y}{3} = \frac{Z-1}{4}$

$(A, B, C)_{DIR} = (+3, +\frac{9}{2}, +6)$

$0, \frac{9}{2}, 7$

$-3, 0, 1$

$\frac{A_0}{A_1} = \frac{6}{3} = \frac{B_0}{B_1} = \frac{2}{1} = \frac{C_0}{C_1} = \frac{12}{6}$

$\therefore$ the two lines are //

shorter : normal to plan $\Pi_1 : \vec{n}_1 = \vec{i} + 2\vec{j} - 2\vec{k}$

$\Pi_2 : \vec{n}_2 = 5\vec{i} - 2\vec{j} - \vec{k}$

direction of line of intersection $\vec{n}_1 \times \vec{n}_2 = -6\vec{i} - 9\vec{j} - 12\vec{k}$

7) A line will intersect a plane, if & only it is not // to the plane

$X - 2Y + Z = 6$

$\frac{2Y}{3} + \frac{5}{3} - \frac{6}{3}Y + Z = \frac{18}{3}$

$-\frac{4Y}{3} + Z = \frac{13}{3}$

$X - \frac{3Z-6}{2} - 2 + Z = 6$

$X - \frac{3Z}{2} - 5 = 6$

$X - \frac{5Z}{2} = 11$

$X = \frac{Z}{2} - 2 + 1$

$\frac{Z}{2} + 2Y + Z = 6$

$\frac{4}{3}Y + Z = \frac{12}{3}$

$-\frac{4Y}{3} + Z = \frac{13}{3}$

$\frac{X-1}{2} = \frac{Y+1}{3}$

$X = \frac{2(Y+1)}{3} + 1$

$Y = \frac{3Z-6}{4} - 1$

Equations are not compatible, implying that solving the equations simultaneously yields nothing. Thus, the line is // to the plane

1a)

$$D = \begin{vmatrix} 3 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 4 & 6 \end{vmatrix} = 36 - 24 - 12 = 0$$

YES

b) $$D = \begin{vmatrix} 2 & -1 & 1 \\ 4 & 1 & -3 \\ 7 & 4 & -9 \end{vmatrix} = -18 + 21 + 16 + 12 + 7 - 36 \neq 0$$

NO

c) $$D = \begin{vmatrix} 4 & 2 & -1 \\ 2 & 1 & 1 \\ 0 & 0 & 2 \end{vmatrix} = 8 - 8 = 0$$

YES

2a) $(\vec{a}-\vec{b})_{DIR} = -(1, 1, -2)$ $\quad (\vec{c}-\vec{d})_{DIR} = -(2, 2, -4) = -(1, 1, -2)$

They are // $\Rightarrow$ not $\perp$

b) $(\vec{a}-\vec{b})_{DIR} = -(2, -1, 2)$ $\quad (\vec{c}-\vec{d})_{DIR} = (1, 0, -4)$

Not //

$2 + 0 - 8 \neq 0$

Not $\perp$

c) $(\vec{a}-\vec{b})_{DIR} = -(5, 3, -2)$ $\quad (\vec{c}-\vec{d})_{DIR} = (1, -1, 1)$

not //

$5 - 3 - 2 = 0$

They are $\perp$

3) $x_1 = 2 + (0)t$ $\quad x_2 = 2 - 5t$

$y_1 = -1 - 2t$ $\quad y_2 = -1 + 3t$

$z_1 = 1 + 3t$ $\quad z_2 = 1 + 2t$

$\frac{0}{-5} \neq \frac{-2}{3} \neq \frac{3}{2} \Rightarrow$ not //

$0 - 6 + 6 = 0 \Rightarrow L_1 \perp L_2$

c) $x_1 = {}^-2 + t$ $\quad x_2 = {}^-2 + 3t$

$Y_1 = 2$ $\quad Y_2 = 2$

$Z_1 = {}^-3 + 6t$ $\quad Z_3 = {}^-3 + 2t$

$$3 + 0 + 12 \neq 0 \Rightarrow L_1 \not\perp L_2$$

b) $X_1 = {}^-5 + 14t$ $\quad X_2 = 4 - t$

$Y_1 = 1 + 2t$ $\quad Y_2 = {}^-7 + 9t$

$Z_1 = {}^-8 + 13t$ $\quad Z_2 = 4 - 3t$

$$^-14 + 18 - 39 \neq 0 \Rightarrow L_1 \not\perp L_2$$

4) $L_1 \left\{ \frac{x_1 - 1}{1} = \frac{z_1}{2} \text{ and } Y_1 = {}^-1 \right\}$

$L_2 \left\{ \frac{x_2 + 2}{+5} = \frac{z_2 - 4}{-10} \text{ and } Y_2 = 0 \right\}$

$$\Longrightarrow L_1 \neq L_2$$

5) $(\vec{b} - \vec{c})_{DIR} = (4, {}^-12, -4)$

$\vec{a} = (3, 1, 5)$ $\quad \vec{d} = (7, {}^-11, 1)$

$$\frac{x - 3}{4} = \frac{Y - 1}{-12} = \frac{z - 5}{-4}$$

B

DE II Test 2 November 2, 199?

Name Bob Marks Box 156

1 If $z = x^2 + xy + y^2 + 3x - 3y + 4$,

(i) Find the equation of the tangent plane and the normal at $P_0(-1, 1, \ldots)$

(ii) Find the point(s) where z attains an extreme value and determine whether it is a maximum, a minimum or a saddle point.

$\frac{\partial z}{\partial x} = 2X + Y + 3 = 2$ $\quad \frac{\partial z}{\partial Y} = X + 2Y - 3 = -2$

$z + 1 = 2(X+1) - 2(Y-1)$

$z + 1 = 2X + 2 - 2Y + 2$

$z - 3 = 2X - 2Y$ → Tangent plane

$\frac{X+1}{2} = \frac{Y-1}{-2} = \frac{z+1}{-1}$ → normal line

$2X + Y = -3$ $\quad D = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$ $\quad D_Y = \begin{vmatrix} 2 & -3 \\ 1 & 3 \end{vmatrix} = 9$

$X + 2Y = 3$

$\frac{\partial^2 z}{\partial X^2} = 2$ not $\frac{\partial^2 z}{\partial Y^2} = 2$ $\quad D_X = \begin{vmatrix} -3 & 1 \\ 3 & 2 \end{vmatrix} = -9$

not sufficient

MINIMUM AT $(-3, 3, -5)$

2 Given $w = xy + yz + zx$,

(i) Obtain dw at $P_0(1, 2, 4)$ if $dx = 1$ $dy = -1$ $dz = 1$

(ii) If the variables x, y, z are related by $x = \frac{1}{2}t$, $y = \frac{1}{2}t^2$, $z = \frac{1}{3}t^3$

obtain $\frac{dw}{dt}$ in two ways.

$\partial w = (Y + Z)\,\partial X$

$\partial w = (X + Z)\,\partial Y$

$\partial w = (Y + X)\,\partial Z$

$dw = 6\,\partial x + 5\,\partial Y + 3\,\partial Z$

$dw = 6 - 5 + 3 = 4$

(OVER)

$w = XY + YZ + ZX$

$X = \frac{1}{2}t$
$Y = \frac{1}{2}t^2$
$Z = \frac{1}{2}t^3$

1) $\frac{\partial w}{\partial X} = Y + Z \quad \frac{\partial X}{\partial t} = \frac{1}{2}$

$\frac{\partial w}{\partial Y} = X + Z \quad \frac{\partial Y}{\partial t} = t$

$\frac{\partial w}{\partial Z} = X + Y \quad \frac{\partial Z}{\partial t} = \frac{3}{2}t^2$

$$\frac{dw}{dt} = \frac{1}{2}(Y+Z) + t(X+Z) + \frac{3}{2}t^2(X+Y) =$$

2) $w = XY + YZ + ZX$

$$= \left(\frac{1}{2}t\right)\left(\frac{1}{2}t^2\right) + \frac{1}{2}(t^2)\left(\frac{1}{2}t^3\right) + \left(\frac{1}{2}t^3\right)\left(\frac{1}{2}t\right)$$

$$= \frac{1}{4}\left[t^3 + t^5 + t^4\right]$$

$$\frac{dw}{dt} = \frac{1}{4}\left[3t^2 + 5t^4 + 4t^3\right] = \frac{3}{4}t^2 + \frac{5}{4}t^4 + t^3 \quad ✓$$

3. For the function $w = xyz$

(i) Compute the directional derivative of $P_0(1,1,1)$ in the direction of the vector $\vec{i}+\vec{j}+\vec{k}$

(ii) Compute the largest value of the directional derivative at P_0

i) $\frac{\partial w}{\partial x} = YZ \quad \frac{\partial w}{\partial Y} = XZ \quad \frac{\partial w}{\partial z} = XY$

$$(\hat{\imath} + \hat{\jmath} + \hat{k}) \cdot (\hat{\imath} + \hat{\jmath} + \hat{k}) = 1 + 1 + 1 = 3$$

DIRECTIONAL DIRIV. $= \frac{3}{\sqrt{3}} = \sqrt{3}$ ✓

$$\frac{dw}{ds} = \frac{\partial w}{\partial x} \cos\alpha + \frac{\partial w}{\partial Y} \cos\beta + \frac{\partial w}{\partial z} \cos\gamma$$

$$N = \cos\alpha i + \cos\beta j + \cos\gamma k$$

MAX FOR $\cancel{\cos = 1}$ no since $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$$1 + 1 + 1 = 3$$

4. Let $y = F(x)$ be given indirectly by the equations $y = f(x,s)$, $g(x,s) = 0$

(i) Prove that $\frac{dy}{dx} = f_x - \frac{g_x}{g_s} f_s$, for all $g_s \neq 0$

(ii) Evaluate $\frac{dy}{dx}$ if $y = x + s$ and $s \sin x + x \sin s = 0$

$$\frac{dY}{dx} = \frac{dY}{dX}$$

this only restates the proble but does not solve it

$$\frac{dY}{dX} = \frac{\partial Y}{\partial X} - \frac{\frac{\partial g}{\partial x}}{\frac{\partial g}{\partial s}} \frac{\partial Y}{\partial S}$$

$$\frac{dY}{dX} = f_x - \frac{g_x}{g_s} f_s$$

$$\frac{dY}{dX} = \cancel{(s\cos X + \sin X) - \frac{s}{X}(\sin X - X\sin X)}$$

$$= \cancel{s\cos X + \sin X} - \frac{s(1-X)\sin X}{X}$$

Bonus Given the surface $z = x^2 + 2y^2 + x - 3y$. Find the equation of the tangent plane that is parallel to $z = 5x + y$

$$\frac{\partial z}{\partial X} = 2X + 1 \qquad \frac{\partial z}{\partial Y} = 4Y - 3$$

$$z - z_0 = (2X_0 + 1)(X - X_0) + (4Y_0 - 3)(Y - Y_0)$$

$$z - z_0 = \frac{\partial z}{\partial X}(X - X_0) + \frac{\partial z}{\partial Y}(Y - Y_0)$$

$$\frac{\partial z}{\partial X} = 5 = 2X_0 + 1 \Rightarrow X_0 = \cancel{1}\,2 \Rightarrow z_0 = \cancel{1}\,5$$

$$\frac{\partial z}{\partial Y} = 1 = 4Y_0 - 3 \Rightarrow Y_0 = 1 \quad \boxed{z - 1 = 5(X - 1) + (Y - 1)}$$

$$z + \cancel{5}\,6 = 5X + Y$$

[illegible] Solutions

1) We use the formulas for the tangent plane and normal line at P_0

$(x-x_0)\frac{\partial z}{\partial x}\Big|_0 + (y-y_0)\frac{\partial z}{\partial y}\Big|_0 = z - z_0$, $\vec{r} = (x_0, y_0, z_0) + t\left[\frac{\partial z}{\partial x}\Big|_0, \frac{\partial z}{\partial y}\Big|_0, -1\right]$

$\frac{\partial z}{\partial x} = 2x+y+3$, $\frac{\partial z}{\partial x}\Big|_0 = 2$, $\frac{\partial z}{\partial y} = x+2y-3$, $\frac{\partial z}{\partial y}\Big|_0 = -2$

Tangent plane : $2(x+1) - 2(y-1) = z+1$ or $2x - 2y - z = -3$

Normal line $\vec{r} = (-1, 1, -1) + t(2, -2, -1)$ or $\frac{x+1}{2} = \frac{y-1}{-2} = \frac{z+1}{-1}$

ii Extreme values for $\frac{\partial z}{\partial x}\Big|_1 = 2x_1 + y_1 + 3 = 0$ and $\frac{\partial z}{\partial y}\Big|_1 = x_1 + 2y_1 - 3 = 0$, or $x_1 = -3$, $y_1 = 3$

From this $z_1 = x_1^2 + x_1 y_1 + y_1^2 + 3x_1 - 3y_1 + 4 = -5$ (P_0 is a point on the given surface)

Consider a point P in the neigborhood of P_1 : $P(x_1 + h, y_1 + k, z)$. Then

$D = z - z_1 = (-3+h)^2 + (-3+h)(3+k) + (3+k)^2 + 3(-3+h) - 3(3+k) + 4 + 5 = h^2 + hk + k^2 = (h + \frac{1}{2}k)^2 + \frac{3}{4}k^2 > 0$

Hence for all P in the negborhood of P_1 : $z > z_1$ and P_1 is a minimum.

2) i) $dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz = (y+z)dx + (x+z)dy + (x+y)dz = 6dx + 5dy + 3dz = 4$

(ii) $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = (y+z)\frac{1}{2} + (x+z)t + (x+y)\frac{3}{2}t^2$

$= \frac{1}{4}(t^2 + t^3) + \frac{1}{2}(t + t^3)t + \frac{3}{4}(t + t^2)t^2 = \frac{3}{4}t^2 + t^3 + \frac{5}{4}t^4$

alternatively : $w = xy + yz + zx = \frac{1}{4}t^3 + \frac{1}{4}t^4 + \frac{1}{5}t^5$, $\frac{dw}{dt} = \frac{3}{4}t^2 + t^3 + \frac{5}{4}t^4$

3) (i) In general the directional derivative at P_0 in the direction of a vector $\vec{u}$ is given by $\hat{u} \cdot \nabla w\big|_0$ where $\hat{u}$ is the unitvector $\frac{\vec{u}}{u}$.

Here $\hat{u} = \frac{1}{\sqrt{3}}(\vec{i} + \vec{j} + \vec{k})$ $\nabla w = yz\vec{i} + xz\vec{j} + xy\vec{k}$, $\nabla w\big|_0 = \vec{i} + \vec{j} + \vec{k}$

the directional derivative is $\hat{u} \cdot \nabla w\big|_0 = \frac{1}{\sqrt{3}}(\vec{i} + \vec{j} + \vec{k}) \cdot (\vec{i} + \vec{j} + \vec{k}) = \frac{3}{\sqrt{3}} = \sqrt{3}$

(ii) The directional derivative takes its largest value in the direction of the gradient. It hence is equal to $|\nabla w| = |\vec{i} + \vec{j} + \vec{k}| = \sqrt{3}$

4) (i) $y = f(x, s) \Rightarrow dy = f_x dx + f_s ds$. $0 = g(x, s) \Rightarrow 0 = g_x dx + g_s ds$

$\Rightarrow ds = -\frac{g_x}{g_s}dx \Rightarrow dy = \left(f_x - \frac{g_x}{g_s}f_s\right)dx \Rightarrow \frac{dy}{dx} = f_x - \frac{g_x}{g_s}f_s$

(ii) $y = x + s \Rightarrow f_x = 1$, $f_s = 1$. $0 = s\sin x + x\sin s \Rightarrow g_x = s\cos x + \sin s$,

$g_s = \sin x + x\cos s \Rightarrow \frac{dy}{dx} = 1 - \frac{s\cos x + \sin s}{x\cos s + \sin x}$

Bonus problem: We want to find the point of contact $P(a,b,c)$

Since $z = x^2+2y^2+x-3y \Rightarrow \frac{\partial z}{\partial x} = 2x+1$, $\frac{\partial z}{\partial y} = 4y-3$. The tangent plane at P has equation $z-c = (2a+1)(x-a) + (4b-3)(y-b)$. This must be parallel to $z = 5x+y$, hence $2a+1 = 5$, $4b-3 = 1 \Rightarrow a = 2$, $b = 1$. Since P is on $z = x^2+2y^2+x-3y \Rightarrow c = a^2+2b^2+a-3b = 5$ and the equation of the tangent plane becomes $z = 5(x-2)+y-1+5 = 5x+y-6$

$\frac{20}{40}$ C

DE II - Test II Nov 25, 1969

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Work problem 1 and three others.

1 Show that $(x^2+y^2)dx - 2xy\,dy$ is not an exact differential, but that $\frac{(x^2+y^2)dx - 2xy\,dy}{x(x^2-y^2)} = \frac{x^2+y^2}{x(x^2-y^2)}dx - \frac{2y}{x^2-y^2}dy$ is exact. Hence solve the equation $(x^2+y^2)dx - 2xy\,dy = 0$. (The function $\frac{1}{x(x^2-y^2)}$ is called an integrating factor)

$N = (X^2+Y^2)$ $M = -2XY$

$\frac{dM}{dX} = -2Y$ $\frac{dN}{dY} = 2Y$

2

$-2Y \neq 2Y$ ∴ not an exact differential

$$\frac{X^2+Y^2}{X(X^2-Y^2)}dx - \frac{2Y}{X^2-Y^2}dY$$

$M = \frac{X^2+Y^2}{X(X^2-Y^2)}$ $N = \frac{-2Y}{X^2-Y^2}$

$M = \frac{1}{X}(X^2+Y^2)(X^2-Y^2)^{-1}$

$$\frac{dM}{dY} = \frac{1}{X}\left[\frac{2Y}{X^2-Y^2} + \frac{2Y(X^2+Y^2)}{(X^2-Y^2)^2}\right]$$

$$= \frac{2Y(X^2-Y^2)+2Y(X^2+Y^2)}{X(X^2-Y^2)^2}$$

$$= \frac{2Y(X^2\cancel{-Y^2}+X^2\cancel{+Y^2})}{X(X^2-Y^2)^2}$$

$$= \frac{4YX}{(X^2-Y^2)^2}$$

4

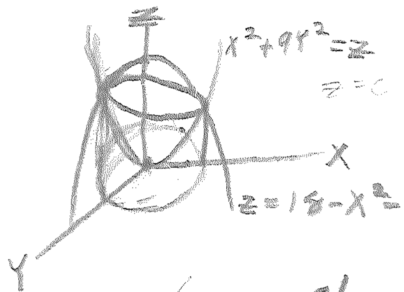
$N = -2Y(X^2-Y^2)^{-1}$

$$\frac{dN}{dX} = \frac{+2Y\,2X}{(X^2-Y^2)^2} = \frac{4XY}{(X^2-Y^2)^2}$$

$$\frac{dM}{dY} = \frac{dN}{dX}$$

$\frac{6}{10}$

2 Find the volume bounded by the ~~ellip~~ elliptic paraboloids $z = x^2 + 9y^2$ and $z = 18 - x^2 - 9y^2$



$9y^2 = z$

$x^2+9y^2 = 18-x^2-9y^2$ $\quad y^2 = 1$

$2x^2 - 18y^2 = 18$

$x = \sqrt{\dfrac{18-18y^2}{2}}$

9

$$\int_{-1}^{1} \int_{-3\sqrt{1-y^2}}^{3\sqrt{1-y^2}} \int_{x^2+9y^2}^{18-x^2-9y^2} dz \; dx\, dy$$

$$\iint 18 - x^2 - 9y^2 - x^2 - 9y^2 \; dx\, dy$$

$$\iint 18 - 2x^2 - 18y^2 \, dx\, dy$$

$$\int_{-1}^{1} 18x - \frac{2x^3}{3} - 18y^2 x \Big]_{-3\sqrt{1-y^2}}^{3\sqrt{1-y^2}} dy$$

$\frac{6}{10}$

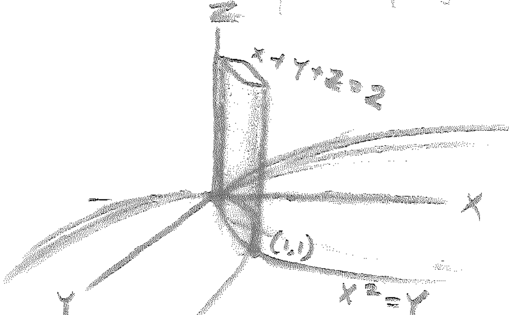
~~$V = \int_{-1}^{1}$~~

3 Find the polar moment of inertia $I_0 = \iint (x^2+y^2)\,dx\,dy$ for a region bounded by the y-axis, the line $y = 2x$ and the line $y = 4$.

$I_0 = \int r^2 \, dm$

$\frac{6}{10}$

4. A region is bounded by the parabolic cylinders $y^2 = x$ and $x^2 = y$ and the planes $z = 0$ and $x + y + z = 2$. Find the volume as a triple integral $\iiint dz\,dy\,dx$. Why is it preferable to first integrate with respect to z? Make a sketch of the region and of its projection on the X-Y plane.



5

$$\int_0^1 \int_{x^2}^{\sqrt{x}} \int_0^{2-x-y} dz\, dy\, dx$$

$$\int_0^1 \int_{x^2}^{\sqrt{x}} (2-x-y)\, dy\, dx$$

$$\int_0^1 \left[(2-x)y - \frac{y^2}{2}\right]_{x^2}^{\sqrt{x}} dx$$

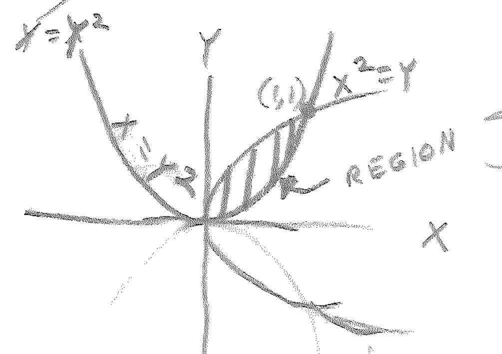
$$\int_0^1 \left[(2-x)\sqrt{x} - \frac{x}{2} + (2-x)x^2 + \frac{x^4}{2}\right]$$

$$\int_0^1 2x^{\frac{1}{2}} - x^{\frac{3}{2}} - \frac{x}{2} - 2x^2 - x^3 + \frac{x^4}{2}$$

$$\frac{4}{3}x^{\frac{3}{2}} - \frac{3}{5}x^{\frac{5}{3}} - \frac{1}{4}x^2 - \frac{2}{3}x^3 - \frac{x^4}{4} + \frac{x^5}{10}\Big|_0$$

$$\frac{4}{3} - \frac{3}{5} - \frac{1}{2} \ldots + \frac{1}{5}$$

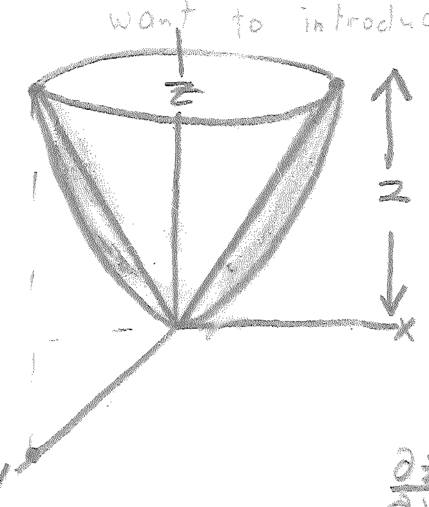
$$\left|\frac{20-12-15}{30}\right| = 7/30$$



3

8/10

5. Find the surface area of the paraboloid of revolution $2z = x^2 + y^2$ which is outside the cone $z^2 = x^2 + y^2$ (You might want to introduce cylinder coordinates)



$$2z = z^2$$
$$z = 2$$

$$x^2 + y^2 = \frac{(x^2+y^2)^2}{4}$$

$$S = \int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \sqrt{1 + \left(x + \frac{y^2}{2}\right)^2 + \left(y + \frac{x^2}{2}\right)^2}\, dx\, dy$$

$$\frac{x^2+y^2}{2}$$

$$\frac{\partial z}{\partial x} = x + \frac{y^2}{2}$$

$$\frac{\partial z}{\partial y} = y + \frac{x^2}{2}$$

$$4 = x^2 + y^2$$
$$x = \sqrt{4 - y^2}$$

8/20

Solutions DE II [illegible]

1 (i) $M_1 = (x^2+y^2)$, $N_1 = -2xy$, $\dfrac{\partial M_1}{\partial y} = 2y$, $\dfrac{\partial N_1}{\partial x} = -2y \neq \dfrac{\partial M_1}{\partial x}$

(ii) $M_2 = \dfrac{x^2+y^2}{x(x^2-y^2)}$, $\dfrac{\partial M_2}{\partial y} = \dfrac{2y}{x(x^2-y^2)} + \dfrac{2xy(x^2+y^2)}{x(x^2-y^2)^2} = \dfrac{4xy}{(x^2-y^2)^2}$

$N_2 = \dfrac{-2y}{x^2-y^2}$, $\dfrac{\partial N_2}{\partial x} = \dfrac{4xy}{(x^2-y^2)^2} = \dfrac{\partial M_2}{\partial x}$

(iii) Let $\dfrac{x^2+y^2}{x(x^2-y^2)}dx - \dfrac{2y}{x^2-y^2}dy = dW = 0$

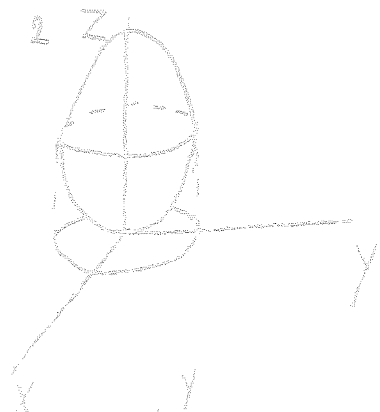
$N_2 = -\dfrac{2y}{x^2-y^2} = \dfrac{\partial W}{\partial y} \Longrightarrow W = -2\displaystyle\int \frac{y\,dy}{x^2-y^2} + \varphi(x) = \ln(x^2-y^2) + \varphi(x)$

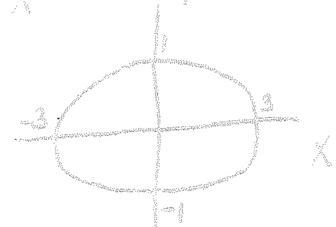
Hence $\dfrac{\partial W}{\partial x} = \dfrac{2x}{x^2-y^2} + \varphi'(x)$ but also $\dfrac{\partial W}{\partial x} = M_2 = \dfrac{x^2+y^2}{x(x^2-y^2)}$. Thus

$\varphi'(x) = \dfrac{x^2+y^2}{x(x^2-y^2)} - \dfrac{2x}{x^2-y^2} = \dfrac{x^2+y^2-2x^2}{x(x^2-y^2)} = -\dfrac{1}{x} \Longrightarrow \varphi(x) = -\ln x + C$

and $W = \ln(x^2-y^2) - \ln x + C = 0$

2





The two paraboloids intersect in

$z = x^2+9y^2 = 18-x^2-9y^2$ or $x^2+9y^2 = 9$, $z = 9$

$$V = \int_{-1}^{1}\int_{-3\sqrt{1-y^2}}^{3\sqrt{1-y^2}}\int_{x^2+9y^2}^{18-x^2-9y^2} dz\,dx\,dy = \int_{-1}^{1}\int_{-3\sqrt{1-y^2}}^{3\sqrt{1-y^2}} (18-2x^2-18y^2)\,dx\,dy =$$

$$\int_{-1}^{1}\left[18x - \tfrac{2}{3}x^3 - 18xy^2\right]_{-3\sqrt{1-y^2}}^{3\sqrt{1-y^2}} dy = \int_{-1}^{1}\left[108(1-y^2)\sqrt{1-y^2} - 36(1-y^2)^{3/2}\right]dy =$$

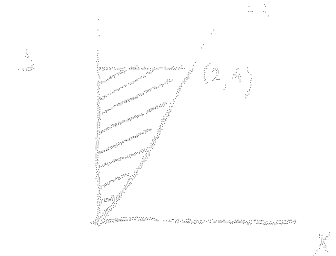
$$72\int_{-1}^{1}(1-y^2)^{3/2}dy. \quad \text{Let } y = \sin\varphi,\ dy = \cos\varphi\, d\varphi$$

$$V = 72\int_{-\pi/2}^{\pi/2}\cos^4\varphi\, d\varphi = 18\int(1+\cos 2\varphi)^2 d\varphi = 18\int(1+2\cos 2\varphi + \cos^2 2\varphi)\,d\varphi$$

$$= 18\int\left(1+2\cos 2\varphi + \tfrac{1}{2} + \tfrac{1}{2}\cos 4\varphi\right)d\varphi = 27\int d\varphi + 36\int \cos 2\varphi\, d\varphi + 9\cdots$$

$$= \left[27\varphi + 18\sin 2\varphi + \tfrac{9}{4}\sin 4\varphi\right]_{-\pi/2}^{\pi/2} = 27\pi$$

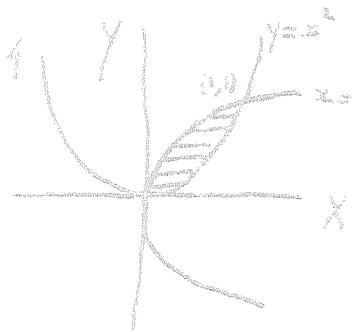
3

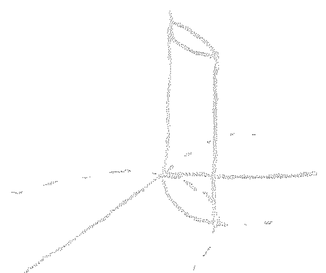


$$I_0 = \int_0^4 \int_0^{y/2} (x^2+y^2)\,dx\,dy = \int_0^4 \left(\tfrac{1}{3}x^3 + xy^2\right)\Big|_0^{y/2} dy$$

$$= \int_0^4 \left(\frac{y^3}{24} + \frac{y^3}{2}\right) dy = \frac{13}{24}\int_0^4 y^3\,dy = \frac{13}{96}\,y^4\Big|_0^4 = \frac{13 \times 256}{96} = \frac{104}{3}$$

4





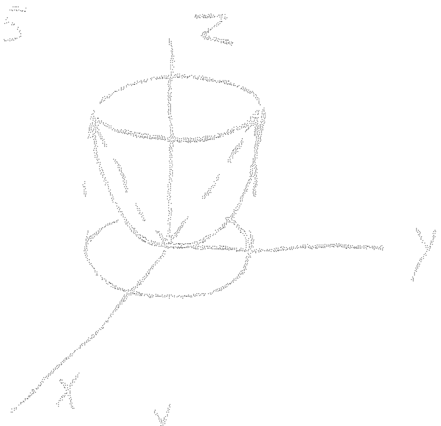
$$V = \int_0^1 \int_{x^2}^{\sqrt{x}} \int_0^{2-x-y} dz\,dy\,dx = \int_0^1 \int_{x^2}^{\sqrt{x}} (2-x-y)\,dy\,dx.$$

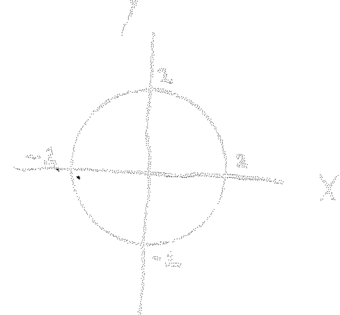
$$= \int_0^1 \left[(2-x)y - \tfrac{1}{2}y^2\right]_{x^2}^{\sqrt{x}} dx = \int_0^1 \left[(2-x)(\sqrt{x}-x^2) - \tfrac{1}{2}x + \tfrac{1}{2}x^4\right] dx$$

$$= \int_0^1 \left(2\sqrt{x} - \tfrac{1}{2}x - x\sqrt{x} - 2x^2 + x^3 + \tfrac{1}{2}x^4\right) dx$$

$$= \tfrac{4}{3}x\sqrt{x} - \tfrac{1}{4}x^2 - \tfrac{2}{5}x^2\sqrt{x} - \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + \tfrac{1}{10}x^5 \Big|_0^1 = \frac{11}{30}$$

5





The cone $z^2 = x^2+y^2$ and the paraboloid $2z = x^2+y^2$ intersect in $z = 2$, $x^2+y^2 = 4$. Furthermore for the paraboloid $\frac{\partial z}{\partial x} = x$, $\frac{\partial z}{\partial y} = y$. The surface area then is $A = \iint \sqrt{x^2+y^2+1}\,dx\,dy$ integrated over the circle $x^2+y^2 \le 4$. Use polar coordinates $x = r\cos\varphi$, $y = r\sin\varphi$. Then $A = \int_0^{2\pi}\int_0^2 \sqrt{r^2+1}\,r\,dr\,d\varphi$, or

$A = 2\pi \int_0^2 \sqrt{r^2+1}\,r\,dr$. Let $r^2+1 = u^2$, $r\,dr = u\,du$

$$A = 2\pi \int u^2\,du = \frac{2}{3}\pi u^3 \Big| = \frac{2}{3}\pi (r^2+1)^{3/2}\Big|_0^2 = \frac{2}{3}\pi(5\sqrt{5}-1)$$

$\frac{26}{60}$

DE I Test IV Dec 9

Name BOB MARKS Box 156

1. Determine whether the following sequences are convergent, and if so find their limit

5 a) $a_n \left\{ \frac{1+n^3}{2-3n^3} \right\} \rightarrow \frac{3n^2}{-9n^2}$ LAHOSPAL'S RULE $\lim_{n\to\infty} a_n = -\frac{1}{3}$

5 b) $\{2^{-n} + \cos n\}$ DIVERGES (cos n DIVERGES)

0 c) $\{\sinh na\}$ $\frac{1}{2}(e^{na} - e^{na}) \rightarrow$

0 d) $\left\{ \frac{2^n+1}{3^n+1} \right\}$

$\frac{10}{20}$

2. Write $.351\,351\,351\ldots$ as a proper fraction

5

$$\begin{array}{r} 1000x = 351.351351\ldots \\ x = .351351\ldots \\ \hline 999x = 351 \end{array}$$

$$x = \frac{351}{999} = \frac{117}{333} = \frac{39}{111} = \frac{13}{37}$$

3. Determine whether the following series are convergent or divergent

0 a) $A = \sum_{n=1}^{\infty} \frac{2n-1}{2n+1}$ $\frac{A}{B} = 2n-1$

$B = \sum_{n=1}^{\infty} \frac{1}{2n+1}$ $\lim \sum_{n=1}^{\infty} 2n-1 = +\infty$, DIVERGENT

How does $\sum \frac{A_n}{B_n}$ diverge $\Rightarrow \sum A_n$ diverge?

5 b) $\sum_{n=1}^{\infty} \frac{1}{2n-1}$

$$\int_1^{\infty} (2n-1)^{-1} = \frac{1}{2} \cancel{\ln} \frac{1}{2n-1} \rightarrow +\infty$$

DIVERGENT

$\frac{20}{35}$

c) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

0 $\quad \cancel{\frac{n!}{2!} = \frac{n!}{2}}$

$\cancel{\lim_{n\to\infty} \frac{n!}{2} = \infty}$

0 d) $A=\sum_{n=1}^{\infty} \frac{n+2^n}{n\,2^n} = \frac{n}{n\,2^n} + \frac{2^n}{n\,2^n} = \frac{1}{2^n} + \frac{1}{n}$

$\cancel{\text{DIVERGENT}}$

$\frac{1}{2^n} \downarrow$ CON $\quad \frac{1}{n} \downarrow$ CON

$\therefore$ CONVERGE

try again next year

$\sum \frac{1}{n}$ div

5 $\quad \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n+1}} < \frac{1}{n^{3/2}}$

5/15

$\frac{1}{n^{3/2}}$ CONV $\rightarrow \frac{1}{n\sqrt{n+1}}$ CONV

poor notation

4 Use the definition of convergence of a series to determine the convergence of $A=\sum_{n=1}^{\infty}\left[\frac{4}{n(n+1)} + \left(\frac{2}{3}\right)^n\right]$

0 $\quad \frac{4}{n(n+1)} = \frac{1}{n} - \frac{1}{n+4}$

$A = \sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{\infty} \frac{1}{n+4} + \left(\sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n\right)\left(\sum_{n=1}^{\infty} 2^n\right)$

CON $\quad$ CON $\quad$ CON $\quad$ DIV

5 Find $\lim_{n\to\infty} 2^n n^2/5^n$ by considering $\sum_{n=1}^{\infty} 2^n n^2/5^n$ LA HOS RLE

0/5 $\quad \dfrac{\frac{2^{n+1}(n+1)^2}{5^{n+1}}}{\frac{2^n n^2}{5^n}} = \frac{\cancel{5^n}\, 2^{n+1}(n+1)^2}{\cancel{2^n}\, n^2\, \cancel{5^{n+1}}} \rightarrow \frac{4(n+1)}{10n} \rightarrow \frac{4}{10} = \frac{2}{5}$

So what!

Bonus $\quad$ Test for convergence a) $\sum_{n=1}^{\infty} \frac{2^n+3^n}{3^n+4^n}$, b) $\sum_{n=1}^{\infty} \frac{2n^2+1}{2n^5-1}$

$\cancel{4}$ $\quad \dfrac{\frac{2^{n+1}+3^{n+1}}{3^{n+1}+4^{n+1}}}{\frac{2^n+3^n}{3^n+4^n}} = \frac{(2^{n+1}+3^{n+1})(3^n+4^n)}{(2^n+3^n)(3^{n+1}+4^{n+1})} = \dfrac{\frac{(2^{n+1}+3^{n+1})(3^n+2^{2n})}{(2^n+3^n)}}{}$

6/25

$$\frac{2^n + 3^n}{3^n + 4^n}$$

$$\frac{2^n}{3^n + 2^{2n}} + \frac{3^n}{3^n + 2^{2n}}$$

$$\frac{(2^{n+1})(3^{n+1} + 2^{2(n+1)})}{\cancel{2^{n}}(3^{n+1} + 2^{2(n+1)})}$$

$$\frac{2(3^n + 2^{2n})}{3^{n+1} + 2}$$

DEF Test IV SOLUTIONS

1a) $\lim\limits_{n\to\infty} \frac{1+n^3}{1-3n^3} = \lim\limits_{n\to\infty} \frac{\frac{1}{n^3}+1}{\frac{1}{n^3}-3} = -\frac{1}{3}$, convergent

b) $2^{-n}+\cos n$ has no limit since $\lim\limits_{n\to\infty} \cos n$ does not exist. Divergent

c) $\sinh na = \frac{1}{2}(e^{na}-e^{-na})$ $\lim\limits_{n\to\infty} e^{-na} = 0$, $\lim\limits_{n\to\infty} e^{na} = \infty$, divergent

d) $\lim\limits_{n\to\infty} \frac{2^n+1}{3^n+1} = \lim\limits_{n\to\infty} \left(\frac{2}{3}\right)^n \frac{1+(\frac{1}{2})^n}{1+(\frac{1}{3})^n} = 0 \times \frac{1+0}{1+0} = 0$

2 $S = .351\,351\cdots$ geometric series, first term $.351$, ratio $.001$

Hence $S = \frac{a}{1-r} = \frac{.351}{1-.001} = \frac{351}{999} = \frac{39}{111} = \frac{13}{37}$

3 a) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{2n-1}{2n+1} = \lim\limits_{n\to\infty} \frac{2-\frac{1}{n}}{2+\frac{1}{n}} = 1 \neq 0$. divergent

b) $a_n = \frac{1}{2n-1}$. Integral test $\int_1^\infty \frac{dx}{2x-1} = \frac{1}{2}\ln(2x-1)\Big|_1^\infty = \infty$. divergent

or compare with harmonic series $\frac{1}{2}\sum_1^\infty \frac{1}{n}$ Since

$\frac{1}{2n-1} - \frac{1}{2n} = \frac{1}{2n(2n-1)} > 0 \therefore a_n > \frac{1}{2n}$, divergence by comparison

c) ratio test $\frac{a_{n+1}}{a_n} = \frac{[(n+1)!]^2}{(2n+2)!}\frac{(2n)!}{(n!)^2} = \left[\frac{(n+1)!}{n!}\right]^2 \frac{(2n)!}{(2n+2)!} = \frac{(n+1)^2}{(2n+2)(2n+1)}$

$\frac{a_{n+1}}{a_n} = \frac{1}{4}\frac{n+1}{n+\frac{1}{2}}$ $\lim\limits_{n\to\infty} \frac{1}{4}\frac{n+1}{n+\frac{1}{2}} = \frac{1}{4} < 1$. convergence

d) $\frac{n+2^n}{n\,2^n} = \frac{1}{2^n}+\frac{1}{n} > \frac{1}{n}$. divergence by comparison with harmonic series

e) $n\sqrt{n}+1 > n\sqrt{n}$, hence $\frac{1}{n\sqrt{n}+1} < \frac{1}{n\sqrt{n}}$ convergence by comparison with p-series for $p = 3/2$

4 By definition a series is convergent if the sequence of partial sums has a limit. Hence consider

$S_N = \sum_{n=1}^{N}\left[\frac{4}{n(n+1)} + \left(\frac{2}{3}\right)^n\right]$. This is a <u>finite</u> sum and so its terms may be rearranged. Hence $S_N = \sum_{n=1}^{N}\frac{4}{n(n+1)} + \sum_{n=1}^{N}\left(\frac{2}{3}\right)^n$

However $\sum_{n=1}^{N}\frac{1}{n(n+1)} = \sum_{n=1}^{N}\frac{1}{n} - \sum_{n=1}^{N}\frac{1}{n+1} = \sum_{n=1}^{N}\frac{1}{n} - \sum_{n=2}^{N+1}\frac{1}{n} = 1 - \frac{1}{N+1}$

and $\sum_{n=1}^{N}\left(\frac{2}{3}\right)^n = \frac{2}{3}\frac{1-\left(\frac{2}{3}\right)^N}{1-\frac{2}{3}} = 2 - 2\left(\frac{2}{3}\right)^N$ (finite geometric series)

Hence $S_N = 4 - \frac{4}{N+1} + 2 - 2\left(\frac{2}{3}\right)^N = 6 - \frac{4}{N+1} - 2\left(\frac{2}{3}\right)^N$

and $\lim_{N\to\infty} S_N = 6$ Hence convergence.

5 Let $a_n = \left(\frac{2}{5}\right)^n n^2$. Consider the series $\sum_{n=1}^{\infty} a_n$ and apply ratio test $\frac{a_{n+1}}{a_n} = \left(\frac{2}{5}\right)^{n+1}(n+1)^2 \Big/ \left(\frac{2}{5}\right)^n n^2 = \frac{2}{5}\left(1+\frac{1}{n}\right)^2$ Hence $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \frac{2}{5} < 1$ and $\sum a_n$ is convergent. In a convergent series $\lim_{n\to\infty} a_n = 0$. Hence $\lim_{n\to\infty}\left(\frac{2}{5}\right)^n n^2 = 0$

Bonus a) ratio test $\frac{a_{n+1}}{a_n} = \frac{2^{n+1}+3^{n+1}}{3^{n+1}+4^{n+1}} \times \frac{3^n+4^n}{2^n+3^n} = \left(\frac{3}{4}\right)^{n+1}\frac{\left(\frac{2}{3}\right)^{n+1}+1}{\left(\frac{3}{4}\right)^{n+1}+1} \times \left(\frac{4}{3}\right)^n \frac{\ldots}{\left(\frac{2}{3}\right)^n \ldots}$

$\frac{a_{n+1}}{a_n} = \frac{3}{4}\,\frac{1+\left(\frac{2}{3}\right)^{n+1}}{1+\left(\frac{3}{4}\right)^{n+1}} \times \frac{1+\left(\frac{3}{4}\right)^n}{1+\left(\frac{2}{3}\right)^n}$ $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \frac{3}{4} < 1$ convergence

b) compare with p-3 series

$\frac{1}{n^3} - \frac{2n^2+1}{2n^5-1} < 0$ does not work. Then try

$\frac{1}{(n-1)^3} - \frac{2n^2+1}{2n^5-1} = \frac{6n^4-7n^3+5n^2-3n}{(n-1)^3(2n^5-1)} > 0$ for $n \geq 2$

Hence convergence by comparison with p-3 series

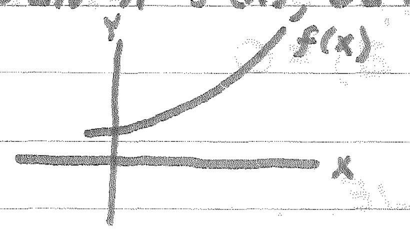
I) POWER SERIES EXPANSIONS OF FUNCTIONS

A) POWER SERIES (DEF)

$$\sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 \ldots$$

B) GIVEN A $f(x)$, DETERMINE BEST FITTING CURVE

1)



2) DIRIVITIVES 1, 2, 3 ... n SHOULD MATCH THE CORRESPONDING DIRIVITIVES OF FUCTION. USING POWER SERIES:

a) $f_n(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$

b) $f_n'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \ldots + n a_n x^{n-1}$

c) $f_n''(x) = 2a_2 + 3 \cdot 2 a_3 x + \ldots + n(n-1) a_n x^{n-2}$

$\vdots$

d) $f_n^n(x) = n!\, a_n$

3) a) IN 2), SUBSTITUTE $x = 0$, & COMPUTE $x = 0$ IN ORIGINAL $f(x)$.

b) DETERMINE COEFFICIENTS $a_0, a_1, \ldots$

4) FOR EXAMPLE, COMUTE $f(x) = e^x$

a) $f(x) = e^x \qquad f(0) = 1$

$f'(x) = e^x \qquad f'(0) = 1$

$\vdots \qquad \vdots$

$f^n(x) = 1 \qquad f^n(0) = 1$

b) $\therefore e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$

$$\left(a_0 = \frac{f(0)}{1},\ a_1 = \frac{f'(0)}{1},\ a_2 = \frac{f''(0)}{2!} + \ldots + \frac{f^n(0)}{n!} \right.$$

C) TAYLOR SERIES

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots \frac{f^{(n)}(a)}{n!}(x-a)^n$$

II) TAYLOR SERIES WITH REMAINDER

A) THE REMAINDER, R_n

$$R_n(x,a) = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt$$

1) IFF $\lim_{n\to\infty} R_n(x,a) = 0$

2) THE ABOVE IMPLIES:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^k(a)}{k!}(x-a)^k$$

B) TAYLOR'S THEOROM

1) LET f BE A FUNCTION OF x THAT IS, ALONG WITH IT'S $n+1$ DIRIVITIVES, CONTINUOUS, ON AN INTERVAL CONTAINING a AND X.

2) $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 \dots + \frac{f^n(a)}{n!}(x-a)^n + R_n(x,a)$

3) $R_n(x,a) = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt$

C) ESTIMATION OF THE REMAINDER

1) THEOROM - IF $g(t)$ AND $h(t)$ ARE CONTINUOUS AT $a \leq t \leq b$, THEN THERE IS A NUMBER c BETWEEN a & b SUCH THAT

$$\int_a^b g(t)(h(t))\,dt = g(c)\int_a^b h(a)\,dt$$

2) LERANGE'S THEOROM STATES:

a) $R_n(x,a) = f^{(n+1)}(c)\frac{(x-a)^{n+1}}{(n+1)!}$; $a < c < b$

b) MAY ESTIMATE $f^{(n+1)}(C)$

3) CAUNCHY'S REMAINDER FORM:

a) $R_n(x,a) = \frac{(x-c')^n(x-a)}{n!} f^{n+1}(c')$

b) $a < c' < x$

III) INTERMEDIATE FORMS

A) THE INDETEMINATE FORM 0/0

1) CONSIDER $\lim_{x \to a} \frac{f(x)}{g(x)}$ AT A POINT WHERE BOTH $f(x)$ AND $g(x)$ VANISH

2) SUPPOSE $f(x)$ AND $g(x)$ CAN BE EXPRESSED AS TAYLOR SERIES

a) $f(x) = f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots$

b) $g(x) = g(a) + g'(a)\cdot(x-a) + \frac{g''(a)}{2!}(x-a)^2 + \ldots$

c) BOTH CONVERGE, RESPECTIVELY, IN SOME INTERVAL, $|x-a| < \delta$, WHERE $\delta > 0$

3) EXAMPLE: FIND $\lim_{x \to 1} \frac{\ln x}{x-1} \to \frac{0}{0}$

a) $f(x) = \ln x$ AND $g(x) = x-1$

b) TAYLOR SERIES FOR EXPAN. OF $\ln x$:

$\ln x = 0 + (x-1) - \frac{1}{2}(x-1)^2 + \ldots$

c) $\frac{\ln x}{x-1} = \frac{(x-1) - \frac{1}{2}(x-1)^2 \ldots}{(x-1)}$

$= 1 - \frac{1}{2}(x-1) \ldots =$

d) $\lim_{x \to 1} \frac{\ln x}{x-1} = -\frac{1}{2}$

4) LAHOPITAL'S RULE:

a) $f(a) = g(a) = 0$, AND $f'(t)/g'(t)$ EXISTS

b) THEN $\lim_{t \to a} \frac{f(t)}{g(t)} = \lim \frac{f'(t)}{g'(t)}$

B) THE INDETERMINATE FORM $\frac{\infty}{\infty}$ AND $0 \cdot \infty$

1) MAY TRANSFORM $\frac{\infty}{\infty}$ AND $0 \cdot \infty$ TO $\frac{0}{0}$ $\left(\frac{1}{\infty} = 0\right)$ AND USE LAHOPITAL'S RULE

2) a) $\frac{\infty_1}{\infty_2} = \frac{1/\infty_2}{1/\infty_1} = \frac{0}{0}$

C) THE INDETERMINATE FORM $\infty - \infty$

1) ATTEMPT ALGEBRAIC MANIPULATION OF $\infty - \infty$ TO $\frac{0}{0}$ OR $\frac{\infty}{\infty}$ AND USE LAHOPITAL'S RULE

2) MAY JUST MESS AROUND WITH IT ALGEBRAICALLY: EX)

a) FIND $\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$

b) $\frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x \sin x}$

c) $\frac{f'(x)}{g'(x)} =$

LIMIT GOES TO 0

D) INDETERMINATE FORMS: 0^0, 1^∞, AND ∞^0

1) THESE ARISE WHEN $y = f(t)^{g(t)}$ WHEN

a) $f(a) = g(a) = 0$

b) $f(a) = 1$; $\lim_{t \to a} g(t) = \infty$

c) $g(a) = 0$; $\lim_{x \to a} f(t) = \infty$

2) MAY TAKE $\ln y$, AND ANOTHER INDETERMINATE FORM ARISES:

a) $0 \cdot (-\infty)$

b) $\infty \cdot 0$

3) EX) FIND $\lim_{h \to 0} (1+h)^{\frac{1}{h}}$

a) let $y = (1+h)^{1/h}$

b) $\ln y = \frac{1}{h} \ln(1+h) \to \frac{0}{0}$ AS $h \to 0$

c) MAY USE LA HOPITAL'S RULE

d) ANSWER IS 1

IV) THE LAPLACE TRANSFORM

A) DEFINITION:

1) $\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\,dt = f(s)$

2) S IS THE PARAMETER

3) $\mathcal{L}$ IS A LINEAR OPERATOR

$\therefore \mathcal{L}\{c_1F_1(t) + c_2F_2(t)\} = c_1\mathcal{L}\{F_1(t)\} + c_2\mathcal{L}\{F_2(t)\}$

B) TRANSFORMS OF ELEMENTRY FUNCTIONS

1) $\mathcal{L}\{e^{kt}\}$

a) $\mathcal{L}\{e^{kt}\} = \int_0^\infty e^{-st} e^{kt}\,dt$

$= \int_0^\infty e^{-(s-k)t}\,dt$

b) $s > K$ IF INTEGRAL CONVERGES

c) $\mathcal{L}\{e^{kt}\} = \left[\dfrac{-e^{-(s-k)t}}{s-k}\right]_0^\infty$

d) $\mathcal{L}\{e^{kt}\} = \dfrac{1}{s-k} \qquad s > k$

e) $\mathcal{L}\{e^{kt}\} = \mathcal{L}\{e^0\} = \dfrac{1}{s} \qquad \begin{matrix} s > k \\ k = 0 \end{matrix}$

2) $\mathcal{L}\{\sin kt\}$

a) $\mathcal{L}\{\sin kt\} = \int_0^\infty e^{-st} \sin kt\,dt$

$= \left[\dfrac{e^{-st}(-s\sin kt - k\cos kt)}{s^2+k^2}\right]_0^\infty$

b) $\mathcal{L}\{\sin kt\} = \dfrac{k}{s^2+k^2}$

3) $\mathcal{L}\{\cos kt\}$ (ANALOGY TO 2 YIELDS):

$\mathcal{L}\{\cos kt\} = \dfrac{s}{s^2+k^2}$

4) $\mathcal{L}\{t^n\} \qquad n = 1, 2, 3, 4, \ldots$

a) BY INTEGRATING BY PARTS OF $\mathcal{L}\{t^n\} = \int_0^\infty e^{-st} t^n\,dt$

$\mathcal{L}\{t^n\} = \left[\dfrac{-t^n e^{-st}}{s}\right]_0^\infty + \dfrac{n}{s}\int_0^\infty e^{-st} t^{n-1}\,dt$

b) FOR $S > 0$

1) $\int_0^\infty e^{-st} t^n dt = \frac{n}{s} \int_0^\infty e^{-st} t^{n-1} dt$

2) OR $\mathcal{L}\{t^n\} = \frac{n}{s} \mathcal{L}\{t^{n-1}\}$

c) THIS PROVIDES A SERIES, WHICH YIELDS:

$\mathcal{L}\{t^n\} = n!/s^{n+1} \qquad s > 0$

5) FOR $H(t)$ WHERE:

$H(t) = 5 \quad t > 4 \; ; \; H(t) = t \qquad 0 < t < 4$

a) $\mathcal{L}\{H(t)\} = \int_0^\infty e^{-st} H(t) dt$

$= \int_0^4 e^{-st} t \, dt + \int_4^\infty e^{-st} 5 \, dt$

b) $\mathcal{L}\{H(t)\} = \left[-\frac{t}{s} e^{-st} - \frac{1}{s^2} e^{-st} \right]_0^4 + \left[-\frac{5}{s} e^{-st} \right]_4^\infty$

c) $\mathcal{L}\{H(t)\} = \frac{1}{s^2} + \frac{e^{-4s}}{s} - \frac{e^{-4s}}{s^2}$

V) TRANSFORMING INITIAL VALUE PROBLEMS

A) EX. OF APPYING $\mathcal{L}$ TO DIFFER. EQ.

1) $\frac{dY}{dt} = e^{2t}$; $Y(0) = \frac{1}{2}$

2) $\mathcal{L}\left\{\frac{dY}{dt}\right\} = \mathcal{L}\{e^{2t}\}$

BUT WHAT IS $\mathcal{L}\left\{\frac{dY}{dt}\right\}$

3) BY DEFINITION:

$$\mathcal{L}\left\{\frac{dY}{dt}\right\} = \int_0^\infty e^{-st}\frac{dY}{dt}\,dt$$

4) INTEGRATING BY PARTS YIELDS:

$$\mathcal{L}\left\{\frac{dY}{dt}\right\} = \left[Ye^{-st}\right]_0^\infty + s\int_0^\infty e^{-st}\,Y(t)\,dt$$

5) IF $Ye^{-st} \to 0$ AS $t \to \infty$:

$$\mathcal{L}\left\{\frac{dY}{dt}\right\} = sL\{Y(t)\} - Y(0)$$

6) $\therefore \mathcal{L}\{Y(t)\} = \frac{1}{2}\left(\frac{1}{(s-2)}\right)$; $s > 2$

7) $Y(t) = \frac{1}{2}e^{2t}$

B) SIMILARLY: $\mathcal{L}\left\{\frac{d^2Y}{dt^2}\right\} = sL\left\{\frac{dY}{dt}\right\} - Y(0)$

1) $\mathcal{L}\left\{\frac{d^2Y}{dt^2}\right\} = s\left[s\mathcal{L}\{Y(t)\} - Y(0)\right] - Y'(0)$

2) OR $\mathcal{L}\left\{\frac{d^2Y}{dt^2}\right\} = s^2\mathcal{L}\{Y(t)\} - sY(0) - Y'(0)$

C) EX: $Y'' + Y = 0$; $Y(0) = 0$; $Y'(0) = 1$

1) $\mathcal{L}\{Y'' + Y\} = 0 \Rightarrow \mathcal{L}\{Y''\} + \mathcal{L}\{Y\} = 0$

2) $s^2\mathcal{L}\{Y(t)\} - 1 + \mathcal{L}\{Y(t)\} = 0$

3) SOLVING FOR $\mathcal{L}\{Y(t)\}$ YIELDS:

$$\mathcal{L}\{Y(t)\} = 1/(s^2+1)$$

4) OR $Y(t) = \sin t$

VI) INVERSE TRANSFORMS ($\mathcal{L}^{-1}$)

A) DEFINITION

1) IF $\mathcal{L}\{F(t)\} = f(s)$

2) THEN $F(t) = \mathcal{L}^{-1}\{f(s)\}$

B) THEOREM:

$$\mathcal{L}^{-1}\{c'f_1(s) + c_2 f_2(s)\} = c_1 \mathcal{L}^{-1}\{f_1(s)\} + c_2 \mathcal{L}^{-1}\{f_2(s)\}$$

C) THEOREM:

$$\mathcal{L}^{-1}\{f(s)\} = e^{-at}\mathcal{L}^{-1}\{f(s-a)\}$$

VII) TABLE OF SOME BASIC LAPLACE TRANSFORMS

A) $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$; $s > a$

B) $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$; $s > 0$

C) 1) $\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}$; $s > 0$

2) $\mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2}$; $s > 0$

D) 1) $\mathcal{L}\{\sinh kt\} = \frac{k}{s^2-k^2}$; $s > k$

2) $\mathcal{L}\{\cosh kt\} = \frac{s}{s^2-k^2}$; $s > k$

E) $\mathcal{L}\{k\} = \frac{k}{s}$

F) 1) $\mathcal{L}\{\frac{dy}{dt}\} = s\mathcal{L}\{y(t)\} + {}^{-}y(0)$

2) $\mathcal{L}\{\frac{d^2y}{dt^2}\} = s^2\mathcal{L}\{y(t)\} - sy(0) - y'(0)$

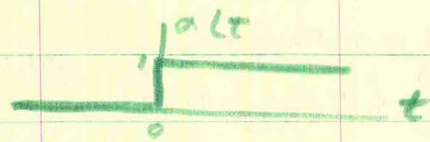
VIII) A STEP FUNCTION

A) DEFINE FUNCTION $\alpha(t)$ BY

1) $\alpha(t) = 0 \quad t < 0$

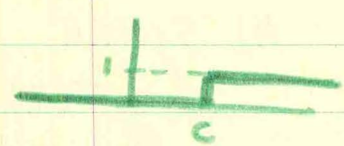
$ = 1 \quad t > 0$

2) GRAPHICALLY:



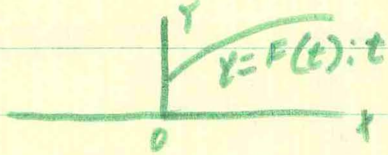
B) $\alpha(t-c) = 0 \quad t < c$

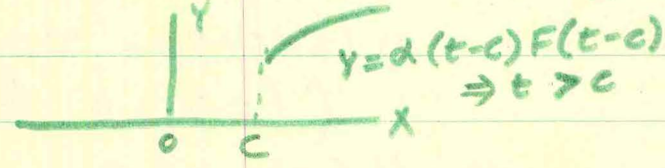
$ = 1 \quad t > c$



C) MAY BE USED TO TRANSLATE GRAPHS

1)





2) $\mathcal{L}$ OF TWO ARE RELATED

a) $\mathcal{L}\{\alpha(t-c)F(t-c)\} = \int_0^\infty e^{-st}\alpha(t-c)F(t-c)dt$

$= \int_c^\infty e^{-st}F(t-c)dt$

b) $t - c = v \Rightarrow$

$\mathcal{L}\{\alpha(t-c)F(t-c)\} = \int_0^\infty e^{-s(c+v)}F(v)dv$

$= e^{-cs}\int_0^\infty e^{-sv}F(v)dv$

c) SUCH AN INTEGRAL IS INDEPENDENT OF THE VARIABLE OF INTEGRATION

$\therefore \int_0^\infty e^{-sv}F(v)dv = \int_0^\infty e^{-st}F(t)dt$

$= \mathcal{L}\{F(t)\} = f(s)$

d) ALSO:

$\mathcal{L}\{\alpha(t-c)F(t-c)\} = e^{-cs}\mathcal{L}\{F(t)\}$

$= e^{-cs}f(s)$

e) THEOREM: IF $\mathcal{L}^{-1}\{f(s)\} = F(t)$; $c \geq 0$; $F(t)$ HAS VALUES FOR $-c < t < 0$, THEN

$\mathcal{L}^{-1}\{e^{-cs}f(s)\} = F(t-c)\,\alpha(t-c)$

IX) PERIODIC FUNCTIONS

A) ~~THEOROM~~ SUPPOSE $F(t+\omega) = F(t)$

1) $\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\,dt$

2) MAY BE WRITTEN AS A SUM OF INTEGRALS

$$\mathcal{L}\{F(t)\} = \sum_{n=0}^{\infty} \int_{n\omega}^{(n+1)\omega} e^{-st} F(t)\,dt$$

3) LET $t = \beta + n\omega$

$$\mathcal{L}\{F(t)\} = \sum_{n=0}^{\infty} \int_0^{\omega} e^{-sn\omega - s\beta} F(\beta + n\omega)\,d\beta$$

4) SINCE $F(\beta + n\omega) = F(\beta)$

$$\mathcal{L}\{F(t)\} = \sum_{n=0}^{\infty} e^{-sn\omega} \int_0^{\omega} e^{-s\beta} F(\beta)\,d\beta$$

5) $\sum_{n=0}^{\infty} e^{-sn\omega} = \sum_{n=0}^{\infty} (e^{-s\omega})^n = \dfrac{1}{1-e^{-s\omega}}$

6) LEADING TO A NEAT THEOREM

$$\mathcal{L}\{F(t)\} = \frac{\int_0^{\omega} e^{-s\beta} F(\beta)\,d\beta}{1 - e^{-s\omega}}$$

B) $H(t)$ HAS A PERIOD $= 2c$

$H(t) = 0$ THROUGHOUT RT. HALF OF EACH PERIOD OR

1) $H(t+2c) = H(t)$

$$\begin{aligned} H(t) &= g(t) & 0<t<c \\ &= 0 & c<t<2c \end{aligned}$$

2) $\mathcal{L}\{H(t)\} = \dfrac{\int_0^{c} e^{-s\beta} g(\beta)\,d\beta}{1 - e^{-2cs}}$

C) EXAMPLE 1:

FIND $\mathcal{L}\{\psi(t,c)\}$ WHERE:

$$\begin{aligned} \psi(t,c) &= 1 & 0<t<c \\ &= 0 & c<t<2c \end{aligned}$$

$$\psi(t+2c,c) = \psi(t,c)$$

X) LINEAR EQUATIONS AND POWER SERIES

A) CONSIDER HOMOGENEOUS LINEAR EQUATION:

1) $b_0(x)Y'' + b_1(x)Y' + b_2(x)Y = 0$

a) WHICH MAY BE SIMPLIFIED:

$Y'' + p(x)Y' + q(x)Y = 0$

b) $p(x)$, $q(x)$, ETC. MUST HAVE DENOMINATORS THAT DON'T VANISH AT $X=0$

2) ASSUME THAT A SOLUTION WITH 2 ARBITRARY CONSTANTS CAN BE HAD IN FORM OF AN INFINITE SERIES.

LET $Y(0)=A$; $Y'(0)=B$

3) $Y''(x) = -p(x)Y'(x) - q(x)Y(x)$

$\Rightarrow Y''(0)$ MAY BE COMPUTED DIRECTLY, AS MAY $Y'''(0)$ ETC.

4) WE MAY $\therefore$ USE MACLAURIAN'S FORMULA:

$$Y(x) = Y(0) + \sum_{n=1}^{\infty} Y^{n}(0)\frac{x^n}{n!}$$

B) CONVERGENCE OF POWER SERIES

1) $\sum_{n=0}^{\infty} a_n x^n$ CONVERGES FOR

a) $x=0$ ONLY, OR

b) ALL FINITE X OR

c) INTERVAL $-R<X<R$

2) IF Σ CONVERGES FOR MORE THAN ONE POINT, IT REPRESENTS A $f(x)$ $\forall$

a) $f(x) = \sum_{n=0}^{\infty} a_n x^n$; $-R<x<R$

b) $f'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$; $-R<x<R$

c) $\int_0^x f(Y)\,dY = \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}$; $-R<x<R$

d) THAT IS; THE SERIES INTEGRATES AND DIFFERENTIATES ALONG WITH $f(x)$

C) ORDINARY POINTS & SINGULAR POINTS

1) FOR A LINEAR DIFFERENTIAL EQUATION:

$$b_0(x)Y^n + b_1(x)Y^{n-1} + \dots b_n(x)Y = R(x)$$

a) THE POINT $X = X_0$ IS CALLED AN ORDINARY POINT OF THE EQUATION IF $b_0(x) \neq 0$.

b) THE SINGULAR POINT FOR THIS EQUATION IS $X = X_1$, FOR WHICH $b_0(x) = 0$

D) SOLUTIONS NEAR AN ORDINARY POINT

1) EXAMPLE: SOLVE $Y'' + 4Y = 0$ NEAR THE ORDINATE POINT $X = 0$

a) $Y = \sum_{n=0}^{\infty} a_n X^n$

b) PLUGGING INTO ORIGINAL FORMULA:

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + 4 \sum_{n=0}^{\infty} a_n x^n = 0$$

c) REDEFINE, SO X^{n-2} IS GENERAL TERM

$$\sum_{n=0}^{\infty} n(n-1) a_n X^{n-2} + 4 \sum_{n=2}^{\infty} a_{n-2} X^{n-1}$$

d) MAY THUS ADD THE TWO SUMMATIONS:

$$\sum_{n=2}^{\infty} [n(n-1)a_n + 4a_{n-2}] X^{n-2} = 0$$

($n=2$ BECAUSE FIRST TWO TERMS IN FIRST SERIES $=0$)

e) SINCE EACH COEFFICIENT IN AN INFINITE SERIES THAT CONVERGES IS $0 \Rightarrow$

$$n(n-1)a_n + 4a_{n-2} = 0 \qquad (n \geq 2)$$

f) RESHUFFLING YIELDS:

$$a_n = \frac{-4a_{n-2}}{n(n-1)}$$

g) $\therefore$ $a_2 = \frac{-4a_0}{2}$ $\quad a_3 = \frac{-4a_1}{3\cdot 2}$

$a_4 = \frac{-4a_2}{12}$ $\quad a_5 = \frac{-4a_3}{20}$

$\vdots \qquad \vdots$

$a_{2K} = \frac{-4a_{2K-2}}{2K(2K-1)}$ $\quad a_{2K+1} = \frac{-4a_{2K-1}}{(2K+1)2K}$

h) ERGO; $a_{2K} = \frac{(-1)^K 4^K}{(2K)!} a_0$ FOR $K \geq 1$

BECAUSE

$$a_{2K} = \frac{(-1)^K 4^K}{(2K)!} a_{2K-2} \quad \text{FOR LEFT}$$

i) FOR RIGHT COLUMB:

$$a_{2K+1} = \frac{(-1)^K 4^K a_1}{(2K+1)!} \; ; \text{ FOR } K \geq 1$$

j) SUBSTITUTING INTO $Y = \sum_{n=0}^{\infty} a_n X^n$

$$Y = a_0 + \sum_{K=1}^{\infty} a_{2K} X^{2K} + a_1 X + \sum_{K=1}^{\infty} a_{2K+1} X^{2K+1}$$

k) EXPANDING FURTHER YIELDS:

$$① \; Y = a_0\left[1 + \sum_{K=1}^{\infty} \frac{(-1)^K 4^K X^{2K}}{(2K)!}\right] + a_1\left[X + \sum_{K=1}^{\infty} \frac{(-1)^K 4^K X^{2K+1}}{(2K+1)!}\right]$$

OR

$$② \; Y = a_0\left[1 + \sum_{K=1}^{\infty} \frac{(-1)^K 4^K X^{2K}}{(2K)!}\right] + \frac{1}{2}a_1\left[2X + \sum_{K=1}^{\infty} \frac{(-1)^K (2X)^{2K+1}}{(2K+1)!}\right]$$

l) $\therefore$ $Y = a_0 \cos 2X + \frac{1}{2} a_1 \sin 2X$

XI) SOLUTIONS NEAR REGULAR SINGLE POINTS

A) SUPPOSE $X = X_0$ IS A SINGULAR PT. OF THE EQUATION (REGULAR S. PTS.)

$$b_0(x)Y'' + b_1(x)Y' + b^2(x)Y = 0$$

1) THEN $b_0(x_0) = 0$, SO $b_0 X$ HAS A FACTOR OF $(x - x_0)$ TO SOME POWER

2) REDEFINE FUNCTION:

a) $Y'' + P(x)Y' + q(x)Y = 0$

b) SINCE $P(x)$ AND $q(x)$, AT LEAST ONE HAS IN IT'S DENOMINATOR $(x - x_0)$

3) ASSUMING $p(x)$ AND $q(x)$ HAVE BEEN REDUCED:

a) IF $x = x_0$ IS A SINGULAR POINT IN THE REDEFINED FUNCTION

b) IF DENOMINATOR OF $P(x)$ DOESN'T CONTAIN A FACTOR $(x - x_0)^n$; $n > 1$

c) IF DENOMINATOR OF $q(x)$ DOESN'T CONTAIN A FACTOR $(x - x_0)^n$; $n > 2$

d) THEN $x = x_0$ IS A REGULAR SINGLE POINT OF REDEFINED EQUATION

e) IF $x = x_0$ IS A SINGULAR POINT, BUT NOT A REGULAR SINGLE POINT, IT IS AN IRREGULAR SINGLE POINT.

B) EXAMPLE: CLASSIFY THE SINGULAR POINTS IN THE FINITE PLANE, OF THE EQUATION:

$$x(x-1)^2(x+2)y'' + x^2y' - (x^3+2x-1)y = 0$$

1) FOR THIS EQUATION:

a) $p(x) = \dfrac{x}{(x-1)^2(x+2)}$

b) $q(x) = \dfrac{-(x^3+2x-1)}{x(x-1)^2(x+2)}$

2) THE SINGULAR POINTS ARE $x = 0, 1, {}^{-}2$

a) $x=0$ IS A REGULAR SINGLER POINT OF ORIGINAL EQUATION

b) $x=1$ IS AN IRREGULAL SINGULAR POINT

c) $x={}^{-}2$ IS A REG. SINGULAR PT.

C) THE INDICIAL EQUATION

1) LET $x=0$ BE A REGULAR SINGLE POINT OF THE EQUATION: $y'' + p(x)y' + q(x)y = 0$ WHERE P & q ARE RATIONAL FUNCTIONS OF X

a) SINCE $p(x)$ CANNOT HAVE IT'S DENOMINATOR A FACTOR OF x^n; $n > 1$:

$$p(x) = \frac{r(x)}{x}$$

① $r(x)$ IS A RATIONAL FUNCTION OF X

② $r(x)$ EXISTS AT $x=0$

b) $p(x)$ HAS A POWER SERIES EXPANTION

$$p(x) = \frac{p_0}{x} + p_1 + p_2x + p_3x^2 + \dots$$

c) $\therefore q(x) = \dfrac{q_0}{x^2} + \dfrac{q_1}{x} + q_2 + q_3x + q_4x^2 + \dots$

d) IT IS REASONABLE THAT

$$y = \sum_{n=0}^{\infty} a_n x^{n+c} = a_0x^c + a_1x^{c+1} + x^{c+2} \dots$$

FOR PROPERLY CHOSEN C AND a_n'S

e) IF WE PUT SERIES FOR Y, $p(x)$ AND $q(x)$ INTO $Y'' + p(x)Y' + q(x) = 0$:

$$\left[c(c-1)a_0x^{c-2} + (1+c)c\,c_1x^{c-1} + (2+c)(1+c)a_2x^c + \ldots + \left[\frac{p_0}{x} + p_1 + p_2x + \ldots\right]\left[ca_0^{c-1} + (1+c)a_1x^c + (2+c)a_2x^{c+1} + \ldots\right] + \left[\frac{q_0}{x^2} + \frac{q_1}{x} + q_2 + \ldots\right]\left[a_0x^c + a_1x^{1+c} + a_2x^{c+2} + \ldots\right] = 0\right]$$

f) PERFORMING INDICATED MULTIPICATIONS:

$$\left[c(c-1)a_0x^{c-2} + (1+c)ca_1x^{c-1} + (2+c)(1+c)a_2x^c + \ldots + p_0ca_0x^{c-2} + [p_0(1+c)a_1 + p_1ca_0]\,x^{c-1} + \ldots + q_0a_0x^{c-2} + [q_0a_1 + q_1a_0]\,x^{c-1} + \ldots = 0\right]$$

g) BECAUSE THE COEFFICIENT OF $x^{c-2} = 0$

$$[c(c-1) + p_0c + q_0]\,a_0 = 0$$

h) SINCE $a_0 \neq 0$

$$c^2 + (p_0 - 1)c + q_0 = 0$$

2) THE ABOVE IS CALLED THE "INDICIAL" EQUATION AT $x = 0$ IF p_0 AN q_0 ARE KNOWN, THE ROOTS OF THIS QUADRATIC ARE $c = c_1$ AND $c = c_2$

D) FORM & VALIDITY OF SOLUTIONS NEAR A REGULAR SINGULAR POINT

1) LET $x = 0$ BE A R.S.P. IN $Y'' + p(x)Y' + q(x)Y = 0$

2) $Y = A\sum_{n=0}^{\infty} a_nx^{n+c_1} + B\sum_{n=0}^{\infty} b_nx^{n+c_2}$

3) OR $Y = (A + B\ln x)\sum_{n=0}^{\infty} a_nx^{n+c_1} + B\sum_{n=0}^{\infty} b_nx^{n+c_2}$

4) THESE INFINITE SERIES WHICH OCCURS IN THE ABOVE FORM CONVERGE IN AT LEAST THE ANNULAR REGIONS BOUNDED BY 2 CIRCLES WITH CENTER AT $x = 0$

VI) INVERSE TRANSFORMS ($\mathcal{L}^{-1}$)

A) DEFINITION

1) IF $\mathcal{L}\{F(t)\} = f(s)$
2) THEN $F(t) = \mathcal{L}^{-1}\{f(s)\}$

B) THEOREM:

$$\mathcal{L}^{-1}\{c_1 f_1(s) + c_2 f_2(s)\} = c_1 \mathcal{L}^{-1}\{f_1(s)\} + c_2 \mathcal{L}^{-1}\{f_2(s)\}$$

C) THEOREM:

$$\mathcal{L}^{-1}\{f(s)\} = e^{-at}\mathcal{L}^{-1}\{f(s-a)\}$$

VII) TABLE OF SOME BASIC LAPLACE TRANSFORMS

A) $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$; $s > a$

B) $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$; $s > 0$

C) 1) $\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}$; $s > 0$

2) $\mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2}$; $s > 0$

D) 1) $\mathcal{L}\{\sinh kt\} = \frac{k}{s^2-k^2}$; $s > k$

2) $\mathcal{L}\{\cosh kt\} = \frac{s}{s^2-k^2}$; $s > k$

E) $\mathcal{L}\{k\} = \frac{k}{s}$

F) 1) $\mathcal{L}\left\{\frac{dy}{dt}\right\} = s\mathcal{L}\{y(t)\} + {}^{-}y(0)$

2) $\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2\mathcal{L}\{y(t)\} - sy(0) - y'(0)$

V) TRANSFORMING INITIAL VALUE PROBLEMS

A) EX. OF APPYING $\mathcal{L}$ TO DIFFER. EQ.

1) $\frac{dY}{dt} = e^{2t}$; $Y(0) = \frac{1}{2}$

2) $\mathcal{L}\left\{\frac{dY}{dt}\right\} = \mathcal{L}\{e^{2t}\}$

BUT WHAT IS $\mathcal{L}\left\{\frac{dY}{dt}\right\}$

3) BY DEFINITION:

$$\mathcal{L}\left\{\frac{dY}{dt}\right\} = \int_0^\infty e^{-st}\frac{dY}{dt}\,dt$$

4) INTEGRATING BY PARTS YIELDS:

$$\mathcal{L}\left\{\frac{dY}{dt}\right\} = \left[Ye^{-st}\right]_0^\infty + s\int_0^\infty e^{-st}Y(t)\,dt$$

5) IF $Ye^{-st} \to 0$ AS $t \to \infty$:

$$\mathcal{L}\left\{\frac{dY}{dt}\right\} = s\mathcal{L}\{Y(t)\} - Y(0)$$

6) $\therefore \mathcal{L}\{Y(t)\} = \frac{1}{2}\left(\frac{1}{(s-2)}\right)$; $s > 2$

7) $Y(t) = \frac{1}{2}e^{2t}$

B) SIMILARLY: $\mathcal{L}\left\{\frac{d^2Y}{dt^2}\right\} = s\mathcal{L}\left\{\frac{dY}{dt}\right\} - Y'(0)$

1) $\mathcal{L}\left\{\frac{d^2Y}{dt^2}\right\} = s\left[s\mathcal{L}\{Y(t)\} - Y(0)\right] - Y'(0)$

2) OR $\mathcal{L}\left\{\frac{d^2Y}{dt^2}\right\} = s^2\mathcal{L}\{Y(t)\} - sY(0) - Y'(0)$

C) EX: $Y'' + Y = 0$; $Y(0) = 0$; $Y'(0) = 1$

1) $\mathcal{L}\{Y'' + Y\} = 0 \Rightarrow \mathcal{L}\{Y''\} + \mathcal{L}\{Y\} = 0$

2) $s^2\mathcal{L}\{Y(t)\} - 1 + \mathcal{L}\{Y(t)\} = 0$

3) SOLVING FOR $\mathcal{L}\{Y(t)\}$ YIELDS:

$$\mathcal{L}\{Y(t)\} = 1/(s^2+1)$$

4) OR $Y(t) = \sin t$

b) FOR $s > 0$

1) $\int_0^\infty e^{-st} t^n \, dt = \frac{n}{s} \int_0^\infty e^{-st} t^{n-1} \, dt$

2) OR $\mathcal{L}\{t^n\} = \frac{n}{s} \mathcal{L}\{t^{n-1}\}$

c) THIS PROVIDES A SERIES, WHICH YIELDS:

$$\mathcal{L}\{t^n\} = n!/s^{n+1} \qquad s > 0$$

5) FOR $H(t)$ WHERE:

$H(t) = 5 \quad t > 4 \; ; \; H(t) = t \qquad 0 < t < 4$

a) $\mathcal{L}\{H(t)\} = \int_0^\infty e^{-st} H(t) \, dt$

$= \int_0^4 e^{-st} t \, dt + \int_4^\infty e^{-st} 5 \, dt$

b) $\mathcal{L}\{H(t)\} = \left[-\frac{t}{s} e^{-st} - \frac{1}{s^2} e^{-st} \right]_0^4 + \left[-\frac{5}{s} e^{-st} \right]_4^\infty$

c) $\mathcal{L}\{H(t)\} = \frac{1}{s^2} + \frac{e^{-4s}}{s} - \frac{e^{-4s}}{s^2}$

IV) THE LAPLACE TRANSFORM

A) DEFINITION:

1) $\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\,dt = f(s)$

2) S IS THE PARAMETER

3) $\mathcal{L}$ IS A LINEAR OPERATOR

$\therefore \mathcal{L}\{c_1F_1(t) + c_2F_2(t)\} = c_1\mathcal{L}\{F_1(t)\} + c_2\mathcal{L}\{F_2(t)\}$

B) TRANSFORMS OF ELEMENTRY FUNCTIONS

1) $\mathcal{L}\{e^{kt}\}$

a) $\mathcal{L}\{e^{kt}\} = \int_0^\infty e^{-st}e^{kt}\,dt$

$= \int_0^\infty e^{-(s-k)t}\,dt$

b) $s > k$ IF INTEGRAL CONVERGES

c) $\mathcal{L}\{e^{kt}\} = \left[\frac{-e^{-(s-k)t}}{s-k}\right]_0^\infty$

d) $\mathcal{L}\{e^{kt}\} = \frac{1}{s-k} \qquad s > k$

e) $\mathcal{L}\{e^{kt}\} = \mathcal{L}\{e^{0}\} = \frac{1}{s} \qquad s > k,\ k = 0$

2) $\mathcal{L}\{\sin kt\}$

a) $\mathcal{L}\{\sin kt\} = \int_0^\infty e^{-st}\sin kt\,dt$

$= \left[\frac{e^{-st}(-s\sin kt - k\cos kt)}{s^2+k^2}\right]_0^\infty$

b) $\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}$

3) $\mathcal{L}\{\cos kt\}$ (ANALOGY TO 2 YIELDS):

$\mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2}$

4) $\mathcal{L}\{t^n\} \qquad n = 1, 2, 3, 4, \ldots$

a) BY INTEGRATING BY PARTS

OF $\mathcal{L}\{t^n\} = \int_0^\infty e^{-st}t^n\,dt$

$\mathcal{L}\{t^n\} = \left[\frac{-t^n e^{-st}}{s}\right]_0^\infty + \frac{n}{s}\int_0^\infty e^{-st}t^{n-1}\,dt$

B) THE INDETERMINATE FORM $\frac{\infty}{\infty}$ AND $0 \cdot \infty$

1) MAY TRANSFORM $\frac{\infty}{\infty}$ AND $0 \cdot \infty$ TO $\frac{0}{0}$ $\left(\frac{1}{\infty} = 0\right)$ AND USE LAHOPITAL'S RULE

2) a) $\frac{\infty_1}{\infty_2} = \frac{1/\infty_2}{1/\infty_1} = \frac{0}{0}$

C) THE INDETERMINATE FORM $\infty - \infty$

1) ATTEMPT ALGEBRAIC MANIPULATION OF $\infty - \infty$ TO $\frac{0}{0}$ OR $\frac{\infty}{\infty}$ AND USE LAHOPITAL'S RULE

2) MAY JUST MESS AROUND WITH IT ALGEBRAICALLY: EX)

a) FIND $\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$

b) $\frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x \sin x}$

c) $\cancel{\frac{f(x)}{g(x)}} =$

LIMIT GOES TO 0

D) INDETERMINATE FORMS: 0^0, 1^∞, AND ∞^0

1) THESE ARISE WHEN $Y = f(t)^{g(t)}$ WHEN

a) $f(a) = g(a) = 0$

b) $f(a) = 1$; $\lim_{t \to a} g(t) = \infty$

c) $g(a) = 0$; $\lim_{x \to a} f(t) = \infty$

2) MAY TAKE $\ln Y$, AND ANOTHER INDETERMINATE FORM ARISES:

a) $0 \cdot (-\infty)$

b) $\infty \cdot 0$

3) EX) FIND $\lim_{h \to 0} (1+h)^{\frac{1}{h}}$

a) let $Y = (1+h)^{1/h}$

b) $\ln Y = \frac{1}{h} \ln(1+h) \to \frac{0}{0}$ AS $h \to 0$

c) MAY USE LA HOPITAL'S RULE

d) ANSWER IS 1

3) CAUNCHY'S REMAINDER FORM:

a) $R_n(x,a) = \frac{(x-c')^n(x-a)}{n!} f^{n+1}(c')$

b) $a < c' < x$

III) INTERMEDIATE FORMS

A) THE INDETEMINATE FORM $0/0$

1) CONSIDER $\lim\limits_{x \to a} \frac{f(x)}{g(x)}$ AT A POINT WHERE BOTH $f(x)$ AND $g(x)$ VANISH

2) SUPPOSE $f(x)$ AND $g(x)$ CAN BE EXPRESSED AS TAYLOR SERIES

a) $f(x) = f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots$

b) $g(x) = g(a) + g'(a)\cdot(x-a) + \frac{g''(a)}{2!}(x-a)^2 + \ldots$

c) BOTH CONVERGE, RESPECTIVELY, IN SOME INTERVAL, $|x-a| < \delta$, WHERE $\delta > 0$

3) EXAMPLE: FIND $\lim\limits_{x \to 1} \frac{\ln x}{x-1} \to \frac{0}{0}$

a) $f(x) = \ln x$ AND $g(x) = x-1$

b) TAYLOR SERIES FOR EXPAN. OF $\ln x$:

$$\ln x = 0 + (x-1) - \tfrac{1}{2}(x-1)^2 + \ldots$$

c) $\frac{\ln x}{x-1} = \frac{(x-1) - \frac{1}{2}(x-1)^2 \ldots}{(x-1)}$

$$= 1 - \tfrac{1}{2}(x-1) \ldots$$

d) $\lim\limits_{x \to 1} \frac{\ln x}{x-1} = -\frac{1}{2}$

4) LaHOPITAL'S RULE:

a) $f(a) = g(a) = 0$, AND $f'(t)/g'(t)$ EXISTS

b) THEN $\lim\limits_{t \to a} \frac{f(t)}{g(t)} = \lim \frac{f'(t)}{g'(t)}$

C) TAYLOR SERIES

$$f(x) \cong f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots \frac{f^{(n)}(a)}{n!}(x-a)^n$$

II) TAYLOR SERIES WITH REMAINDER

A) THE REMAINDER, R_n

$$R_n(x,a) = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt$$

1) IFF $\lim_{n\to\infty} R_n(x,a) = 0$

2) THE ABOVE IMPLIES:

$$f(x) = \sum_{K=0}^{\infty} \frac{f^K(a)}{K!}(x-a)^K$$

B) TAYLOR'S THEOROM

1) LET f BE A FUNCTION OF x THAT IS, ALONG WITH IT'S $n+1$ DIRIVITIVES, CONTINUOUS, ON AN INTERVAL CONTAINING a AND X.

2) $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 \dots + \frac{f^n(a)}{n!}(x-a)^n + R_n(x,a)$

3) $R_n(x,a) = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt$

C) ESTIMATION OF THE REMAINDER

1) THEOROM- IF $q(t)$ AND $h(t)$ ARE CONTINUOUS AT $a \leq t \leq b$, THEN THERE IS A NUMBER c BETWEEN a & b SUCH THAT

$$\int_a^b q(t)(h(t))\,dt = q(c)\int_a^b h(a)\,dt$$

2) LERANGE'S THEOROM STATES:

a) $R_n(x,a) = f^{(n+1)}(c)\frac{(x-a)^{n-1}}{(n+1)!}$; $a < c < b$

b) MAY ESTIMATE $f^{(n+1)}(C)$

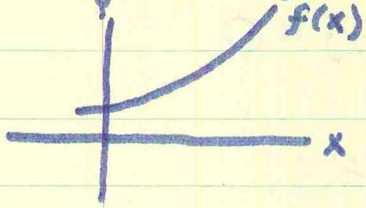
I) POWER SERIES EXPANSIONS OF FUNCTIONS

A) POWER SERIES (DEF)

$$\sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 \ldots$$

B) GIVEN A $f(x)$, DETERMINE BEST FITTING CURVE

1)



2) DIRIVITIVES 1, 2, 3 ... n SHOULD MATCH THE CORRESPONDING DIRIVITIVES OF FUCTION. USING POWER SERIES:

a) $f_n(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$

b) $f_n'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \ldots + n a_n x^{n-1}$

c) $f_n''(x) = 2a_2 + 3 \cdot 2 a_3 x + \ldots + n(n-1) a_n x^{n-2}$

$\vdots$

d) $f_n^n(x) = n! \, a_n$

3) a) IN 2), SUBSTITUTE $x = 0$, & COMPUTE $x = 0$ IN ORIGINAL $f(x)$.

b) DETERMINE COEFFICIENTS $a_0, a_1, \ldots$

4) FOR EXAMPLE, COMUTE $f(x) = e^x$

a) $f(x) = e^x \qquad f(0) = 1$

$f'(x) = e^x \qquad f'(0) = 1$

$\vdots \qquad \vdots$

$f^n(x) = 1 \qquad f^n(0) = 1$

b) $\therefore e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$

$\left(a_0 = \frac{f(0)}{1}, \; a_1 = \frac{f'(0)}{1}, \; a_2 = \frac{f''(0)}{2!} + \ldots + \frac{f^n(0)}{n!} \right.$

	$f(s) = \int_0^\infty e^{-st} F(t)\,dt$	$F(t)$
i	$f(s - a)$	$e^{at} F(t)$
ii	$f(as)$	$\frac{1}{a} F(t/a)$
iii	$e^{-cs} f(s) \qquad c > 0$	$F(t - c)\, H(t - c)$
iv	$sf(s) - F(0)$	$F'(t)$
v	$s^2 f(s) - sF(0) - F'(0)$	$F''(t)$
vi	$s^n f(s) - \sum_{k=1}^{n} s^{n-k} F^{(k-1)}(0)$	$F^{(n)}(t)$
vii	$\frac{1}{s} f(s)$	$\int_0^t F(u)\,du$
viii	$\frac{d^n}{ds^n} f(s)$	$(-t)^n F(t)$
ix	$\int_s^\infty f(u)\,du$	$\frac{1}{t} F(t)$
x	$f(s)g(s)$	$\int_0^t F(u)G(t - u)\,du$
xi	$(1-e^{-Ts})^{-1} \int_0^T e^{-st} F(t)\,dt$	$F(t) = F(t + T)$

.1	$\frac{1}{s}$	$s > 0$	1
2	$\frac{1}{s^{n+1}}$	$n > -1,\ s > 0$	$\frac{t^n}{n!}$
3	$\frac{1}{\sqrt{s}}$	$s > 0$	$\frac{1}{\sqrt{\pi t}}$
4	$\frac{1}{s - a}$	$s > a$	e^{at}
5	$\frac{e^{-cs}}{s}$	$s > 0$	$H(t - c)$
6	$\frac{1}{s^2 + k^2}$	$s > 0$	$\frac{\sin kt}{k}$
7	$\frac{s}{s^2 + k^2}$	$s > 0$	$\cos kt$
8	$\frac{1}{(s^2 + k^2)^2}$	$s > 0$	$\frac{1}{2k^3}(\sin kt - kt \cos kt)$
9	$\frac{s}{(s^2 + k^2)^2}$	$s > 0$	$\frac{t \sin kt}{2k}$
10	$\frac{1}{s^2 - k^2}$	$s > k$	$\frac{\sinh kt}{k}$
11	$\frac{s}{s^2 - k^2}$	$s > k$	$\cosh kt$

Example. Evaluate $L^{-1}\left\{\dfrac{f(s)}{s}\right\}$.

Let $L^{-1}\{f(s)\} = F(t)$. Since $L^{-1}\left\{\dfrac{1}{s}\right\} = 1$, we use Theorem 16 to conclude that

$$L^{-1}\left\{\frac{f(s)}{s}\right\} = \int_0^t F(\beta)\,d\beta.$$

14. *Partial Fractions*

In using the Laplace transform to solve differential equations, we often need to obtain the inverse transform of a rational fraction

$$\frac{N(s)}{D(s)}. \tag{1}$$

The numerator and denominator in (1) are polynomials in s and the degree of $D(s)$ is larger than the degree of $N(s)$. The fraction (1) has the partial fractions expansion used in calculus.* Because of the linearity of the inverse operator L^{-1}, the partial fractions expansion of (1) permits us to replace a complicated problem in obtaining an inverse transform by a set of simpler problems.

Example (a). Obtain $L^{-1}\left\{\dfrac{s^2-6}{s^3+4s^2+3s}\right\}$.

Since the denominator is a product of distinct linear factors, we know that constants A, B, C exist such that

$$\frac{s^2-6}{s^3+4s^2+3s} = \frac{s^2-6}{s(s+1)(s+3)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+3}.$$

Multiplying each term by the lowest common denominator, we obtain the identity

$$s^2 - 6 = A(s+1)(s+3) + Bs(s+3) + Cs(s+1), \tag{2}$$

from which we need to determine A, B, and C. Using the values $s = 0, -1, -3$ successively in (2), we get

$$\begin{aligned} s = 0: &\qquad -6 = A(1)(3),\\ s = -1: &\qquad -5 = B(-1)(2),\\ s = -3: &\qquad 3 = C(-3)(-2),\end{aligned}$$

from which $A = -2$, $B = \frac{5}{2}$, $C = \frac{1}{2}$. Therefore

$$\frac{s^2-6}{s^3+4s^2+3s} = \frac{-2}{s} + \frac{\frac{5}{2}}{s+1} + \frac{\frac{1}{2}}{s+3}.$$

*See, for example, E. D. Rainville, *Unified Calculus and Analytic Geometry*, New York, Macmillan, 1961, pp. 357–364.

Since $L^{-1}\left\{\frac{1}{s}\right\} = 1$ and $L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$, we get the desired result,

$$L^{-1}\left\{\frac{s^2-6}{s^3+4s^2+3s}\right\} = -2 + \tfrac{5}{2}e^{-t} + \tfrac{1}{2}e^{-3t}.$$

Example (*b*). Obtain $L^{-1}\left\{\frac{5s^3-6s-3}{s^3(s+1)^2}\right\}$.

Since the denominator contains repeated linear factors, we must assume partial fractions of the form shown:

$$\text{(3)} \qquad \frac{5s^3-6s-3}{s^3(s+1)^2} = \frac{A_1}{s} + \frac{A_2}{s^2} + \frac{A_3}{s^3} + \frac{B_1}{s+1} + \frac{B_2}{(s+1)^2}.$$

Corresponding to a denominator factor $(x-\gamma)^r$ we must in general, assume r partial fractions of the form

$$\frac{A_1}{x-\gamma} + \frac{A_2}{(x-\gamma)^2} + \cdots + \frac{A_r}{(x-\gamma)^r}.$$

From (3) we get

$$\text{(4)} \quad 5s^3 - 6s - 3 = A_1s^2(s+1)^2 + A_2s(s+1)^2 + A_3(s+1)^2 + B_1s^3(s+1) + B_2s^3,$$

which must be an identity in s. To get the necessary five equations for the determination of A_1, A_2, A_3, B_1, B_2, two elementary methods are popular. Specific values of s can be used in (4), or the coefficients of like powers of s in the two members of (4) may be equated. We employ whatever combination of these methods yields simple equations to be solved for $A_1, A_2, \cdots, B_2$. From (4) we obtain

$$\begin{array}{ll}
s = 0: & -3 = A_3(1), \\
s = -1: & -2 = B_2(-1), \\
\text{coeff. of } s^4: & 0 = A_1 + B_1, \\
\text{coeff. of } s^3: & 5 = 2A_1 + A_2 + B_1 + B_2, \\
\text{coeff. of } s: & -6 = A_2 + 2A_3.
\end{array}$$

The above equations yield $A_1 = 3,\ A_2 = 0,\ A_3 = -3,\ B_1 = -3,\ B_2 = 2$. Therefore we find that

$$L^{-1}\left\{\frac{5s^3-6s-3}{s^3(s+1)^2}\right\} = L^{-1}\left\{\frac{3}{s} - \frac{3}{s^3} - \frac{3}{s+1} + \frac{2}{(s+1)^2}\right\}$$
$$= 3 - \tfrac{3}{2}t^2 - 3e^{-t} + 2te^{-t}.$$

Example (*c*). Obtain $L^{-1}\left\{\frac{16}{s(s^2+4)^2}\right\}$.

Since quadratic factors require the corresponding partial fractions to have linear numerators, we start with an expansion of the form

$$\frac{16}{s(s^2+4)^2} = \frac{A}{s} + \frac{B_1 s + C_1}{s^2+4} + \frac{B_2 s + C_2}{(s^2+4)^2}.$$

From the identity

$$16 = A(s^2+4)^2 + (B_1 s + C_1)s(s^2+4) + (B_2 s + C_2)s$$

it is not difficult to find the values $A = 1$, $B_1 = -1$, $B_2 = -4$, $C_1 = 0$, $C_2 = 0$. We thus obtain

$$L^{-1}\left\{\frac{16}{s(s^2+4)^2}\right\} = L^{-1}\left\{\frac{1}{s} - \frac{s}{s^2+4} - \frac{4s}{(s^2+4)^2}\right\}$$
$$= 1 - \cos 2t - t \sin 2t.$$

It is possible to obtain formulas for the partial fractions expansion of the rational fractions being treated in this section. Such formulas are useful in theory and not particularly inefficient in practice. The elementary techniques above, if used intelligently, are efficient in numerical problems and are the only partial fractions methods presented in this short treatment of the subject.

EXERCISES

In Exs. 1-10, find an inverse transform of the given $f(s)$.

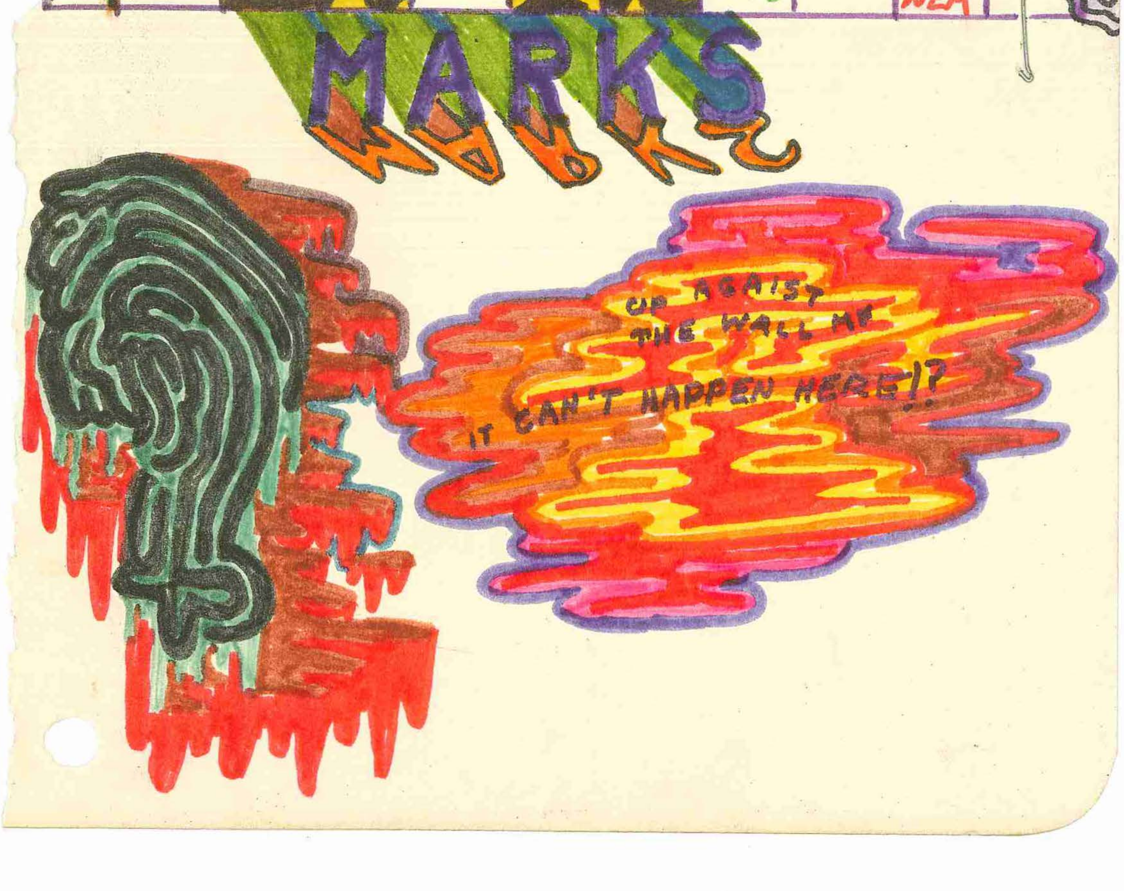
1. $\dfrac{1}{s^2+as}$. *Ans.* $\dfrac{1}{a}(1 - e^{-at})$.
2. $\dfrac{s+2}{s^2-6s+8}$. *Ans.* $3e^{4t} - 2e^{2t}$.
3. $\dfrac{2s^2+5s-4}{s^3+s^2-2s}$. *Ans.* $2 + e^t - e^{-2t}$.
4. $\dfrac{2s^2+1}{s(s+1)^2}$. *Ans.* $1 + e^{-t} - 3te^{-t}$.
5. $\dfrac{4s+4}{s^2(s-2)}$. *Ans.* $3e^{2t} - 3 - 2t$.
6. $\dfrac{1}{s^3(s^2+1)}$. *Ans.* $\frac{1}{2}t^2 - 1 + \cos t$.
7. $\dfrac{5s-2}{s^2(s+2)(s-1)}$. *Ans.* $t - 2 + e^t + e^{-2t}$.
8. $\dfrac{1}{(s^2+a^2)(s^2+b^2)}$, $a^2 \neq b^2$, $ab \neq 0$. *Ans.* $\dfrac{b \sin at - a \sin bt}{ab(b^2-a^2)}$.
9. $\dfrac{s}{(s^2+a^2)(s^2+b^2)}$, $a^2 \neq b^2$, $ab \neq 0$. *Ans.* $\dfrac{\cos at - \cos bt}{b^2-a^2}$.
10. $\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$, $a^2 \neq b^2$, $ab \neq 0$. *Ans.* $\dfrac{a \sin at - b \sin bt}{a^2-b^2}$.
11. Obtain the answers to Exs. 9 and 10 from that for Ex. 8.
12. Use equation (8), page 15 and the convolution. Theorem 16, to obtain

$$L^{-1}\left\{\frac{16}{s(s^2+4)^2}\right\} = \int_0^t (\sin 2\beta - 2\beta \cos 2\beta)\,d\beta,$$

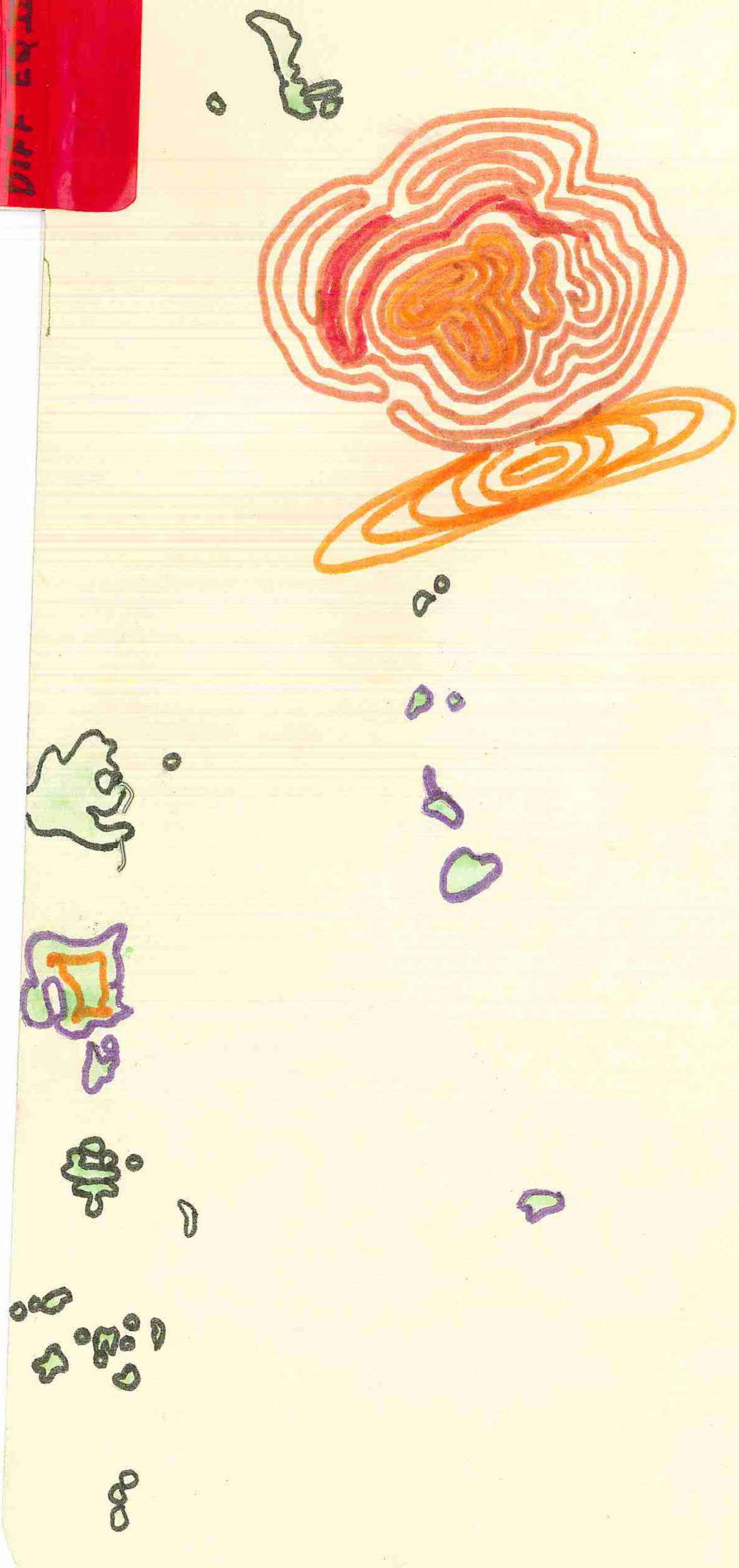
and then perform the integration to check the answer to Example (c), page 32.

DIFF EQ II

	1-7:50	2-8:45	3-9:40	4-10:35	5-11:30	6-12:25	7-1:20	8-2:15	9-3:10	10-4:05
MON	E-SCI A-241 NEM		CONC. OF PWR 004	PHYS. C-124			HUMAN F-205		DIFF EQ A-205	
TUES			DIFF EQ A-205	CONVO			HUM A-121			
WED	E-SCI MOORE F013	CONC OF PWR		PHYS KORT. B-205			HUM A-121	PHYSICS C-04		LAB
THUR		CONC OF PWR	DIFF EQ A-205				HUM A-121	R	J	M
FRI	E SCI			PHYS			DIFF EQ A-205	EL.	SCI D101 NEM	ENCE



DIFF EQ I



15 (3,4)

IF $\lim_{n \to \infty} \frac{a_n}{b_n} \neq 0, \infty$; THEN a_n & b_n

ARE EITHER BOTH CONV. OR DIV.

FIND SIN 5°

AROUND 5°

$\sin x \cong x$

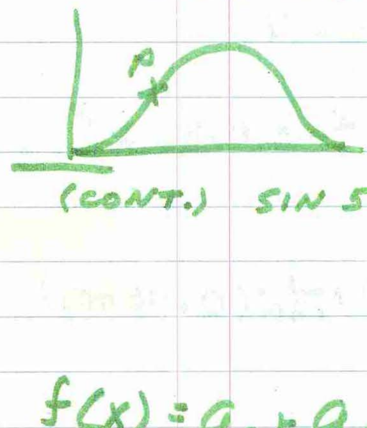
BETTER AROXIMATION:

$\sin x \simeq x + ax^2$

$\sin x \simeq x + ax^2 + bx^3$

∴ TO FIND VALUE OF FUNCTION, APROX. FUNCTION IN POLYNOMIAL:

$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \ldots a_nx^n$



FIND POLY THAT LOOKS LIKE FUCTION AROUND P

(CONT.) SIN 5°

SIMILAR FUNCTION HAS LOTS OF EQUAL DIRIVITIVES

$f(x) = a_0 + a_1x + a_2x^2 \ldots a_nx^n$

$f'(x) = a_1 + 2a_2x + 3a_3x^2 + \ldots na_nx^{n-1}$

$f''(x) = 2a_2 + 6a_3x \ldots n(n-1)x^{n-2}$

$\vdots$

$f(0) = a_0 \quad f'(0) = a_1 \quad f''(0) = 2a_2$

$f'''(0) = 6a_3$

FROM ABOVE

$$f^{IV}(0) = 24\,a_4$$
$$f^{(n)}(0) = n!\,a_n$$

$$\therefore a_n = \frac{f^{(n)}(0)}{n!}$$

DIFFER k TIMES (k IS FINITE)

$$f^{k}(0) = \frac{d^k}{dx^k} a_k x^k = k!\,a_k$$

$$f(x) \simeq f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \frac{1}{6}f'''(0)x^3 \ldots + \frac{1}{k!}f^{k}(0)x^k \ldots + \frac{1}{n!}f^{n}(0)x^n$$

$$f(x) = \sin x \rightarrow f(0) = 0$$
$$f'(x) = \cos x \rightarrow f'(0) = 1$$
$$f''(x) = -\sin x \rightarrow f''(0) = 0$$
$$f'''(x) = -\cos x \rightarrow f'''(0) = -1$$
$$f^{IV}(x) = \sin x$$

$$\therefore \sin x = 0 + x + 0 - \frac{1}{6}x^3 + 0 + \frac{1}{120}x^5 \ldots$$

$$5^\circ \cong .1$$
$$\therefore \sin(.1) \cong .1 - \frac{1}{6}(.001) + \frac{1}{120}(0.00001)$$

TEST FOR FINDING $f(x)$

$$f(x) = \sum_{k=0}^{n} \frac{f^{k}(0)}{k!} x^{k} + R(x,0)$$

rest

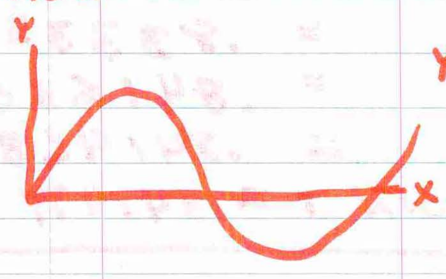
CALLED A MacLAURIN SERIES

$$f(x) = \sum_{k=0}^{n} \frac{f^{k}(a)}{k!} (x-a)^{k} + R_n(x,a)$$

CALLED A TAYLOR SERIES

1-9-70

REVIEW



APROXIMATE SINX;

$$f(x) \cong a_0 + a_1 x + a_2 x^2 \ldots a_n x^n$$

$$f(0) = a_0$$

$$f'(0) = a_1$$

$$f''(0) = 2a_2$$

$$\vdots$$

$$f^{n}(0) = n!\,a_n$$

$$f(x) = f(0) + xf'(0) + \ldots \frac{1}{n!} x^{n} f^{n}(0)$$

$f(x) = \sin x$

$f(x) \cong x$

$f(x) \cong \cancel{2x^3}\ x - \frac{1}{6}x^3$

$f(x) \cong x - \frac{1}{6}x^3 + \frac{1}{120}x^5$

$f(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 \ldots \frac{1}{n!}x^n$

IF $X = .1$ ($5°\ 43'\ 46''$)

FIRST APROX = .1

2 " = .099833

3 " = .099833

IF $X = 1$ ($57°\ 17'\ 45''$)

FIRST APROX = 1.0

2 " = .8333333

3 " = .8416667

4 " = .8414683

IN TABLE, SIN 1 = .841471

MACLAURIN EXPANSION:

$f(x) \simeq f(0) + f'(0) + \frac{1}{2}x^2(f''(0) \ldots \frac{x^n}{n!} f^n(0)$

$f(x) \simeq f(a) + (x-a)f'(a) \ldots \frac{1}{n!}(x-a)^n f^n(a)$

TAYLOR EXPANSION

MACLARIN EX. IS A TAYLOR EXP.

$$x = y - a$$

$$f(y-a) \simeq f(0) + (x-a)\left.\frac{df}{dy}\right|_{y=0} + \frac{1}{2}(y-a)^2\left.\frac{d^2f}{dy^2}\right|_{y=0} + \dots + \frac{1}{n!}(y-a)^n\frac{d^nf}{dy^n}\bigg|_{y=0}$$

$$\left(\frac{df}{dx} = \frac{dx}{dy}\frac{df}{dy} = \frac{df}{dy}\right)$$

$$f(y-a) = F(y)$$

$$(f(0) = F(a)$$

EX) $f(x) = x^2$

$f(y-a) = (y-a)^2$

$F(y) = y^2 - 2ay + a^2$

$$F(y) \simeq F(a) + (y-a)F'(a) + \frac{1}{2}(y-a)^2F''(a) + \dots$$

$$\dots \frac{1}{n!}(y-a)^nF^n(a)$$

$\therefore$ TAYLOR & MACLAURIN SERIES ARE EQUIVILENT

TO GET COSINE SERIES

LET $a = 45°$

$\cancel{f(\tfrac{1}{4}\pi) =}$

$$f(x) \cong f(\tfrac{1}{4}\pi) + (x - \tfrac{1}{4}\pi) f'(\tfrac{1}{4}\pi) + \ldots$$
$$\tfrac{1}{2}(x - \tfrac{1}{4}\pi)^2 f''(\tfrac{1}{4}\pi)$$
$$\cong \tfrac{1}{2}\sqrt{2} + (x - \tfrac{1}{4}\pi)\tfrac{1}{2}\sqrt{2} - \tfrac{1}{2}(x - \tfrac{1}{4}\pi)^2 \tfrac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{2}}{2} + (x - \tfrac{1}{4}\pi)\frac{\sqrt{2}}{2} - \frac{1}{2}(x - \tfrac{\pi}{4})^2 \frac{\sqrt{2}}{2} \ldots$$
$$+ \frac{1}{n!}(x - \tfrac{1}{4}\pi)^n \frac{\sqrt{2}}{2}$$

$$\int_0^x f'(x)\,dx = f(x) - f(0)$$

$$\therefore f(x) = f(0) + \int_0^x f'(x)\,dx$$

1-12-70

$$f(x) \simeq f(a) + (x-a) f'(a) + \ldots \frac{1}{n!}(x-a)^n f^n(a)$$

EX) $\sin x$ $a = 30°$

$$\sin x = \tfrac{1}{2} + (x - \tfrac{\pi}{6})\tfrac{1}{2}\sqrt{3} - \tfrac{1}{2}(x - \tfrac{\pi}{6})^2 \tfrac{1}{2} -$$
$$\tfrac{1}{6}(x - \tfrac{\pi}{6})^3 \tfrac{1}{2}\sqrt{3} \ldots$$
$$\frac{1}{n!}(x - \tfrac{\pi}{6})^n \tfrac{1}{2}\sqrt{3}$$

GOOD FORMULA FOR ANGLES AROUND $30°$, ie $(x - \frac{\pi}{6}) \simeq 0$

THE REMAINDER

$$\int_a^x f'(t)\,dt = f(x) - f(a)$$

$$f(x) = f(a) + \int_a^x f'(t)\,dt$$

$$u = f'(t) \qquad v = t - x$$

$$f(x) = f(a) + (t-x)\, f'(t)\Big|_a^x - \int_a^x (t-x) f''(t)\,dt$$

$$f(x) = f(a) + (x-a)f'(a) - \int_a^x (t-x)\, f''(t)\,dt$$

$$u = f''(t) \quad v = \tfrac{1}{2}(t-x)^2$$

$$= f(a) + (x-a)f'(a) - \tfrac{1}{2}(t-x)^2 f'(t) + \tfrac{1}{2}\int_a^x (t-x)^2 f'''(t)\,dt$$

$$= f(a) + (x-a)f'(a) + \tfrac{1}{2}(x-a)^2 f''(a) + \frac{1}{n!}(x-a)^2 f_a^n + R_n(x,a)$$

$$f(x) = R_n = \frac{1}{n!}\int_a^x (t-x)$$

$$f(x) = R_n = \frac{1}{n!}\int_a^x (t-x)^n f^{(n+1)}(t)\,dt$$

(ALTENATING SIGNS)

EXACTLY EQUAL

$\lim_{n\to\infty} \frac{x^n}{n!} = 0$, FOR FIXED VALUES OF X

CONSIDER $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

USING RATIO TEST

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} \frac{n!}{x^n} =$$

$$\lim_{n\to\infty} \frac{x}{n+1} \quad \therefore$$

$$\lim_{n\to\infty} a_n = 0 \Longrightarrow \lim_{n\to\infty} \frac{x^n}{n!} = 0$$

$$R_n = \frac{1}{n!}\int_a^x (x-t)^n f^{n+1}(t)\,dt$$

EX)

$f(x) = \sin x \qquad -1 \le \sin x \le 1$

R_n MUST LIE BETWEEN -1 AND 1

$$|R_n| \le \frac{1}{n!}\int_0^x (x-t)^n\,dt = \frac{x^{n+1}}{(n+1)!}$$

Pg 641-2 #2,4,5,6,7

1-13-70

INTEGRATION BY PARTS

$$\frac{d}{dt}UV = U\frac{dV}{dt} + V\frac{dU}{dt}$$

$$UV\Big|_{w}^{x} = \int_w^x U\frac{dV}{dt}dt + \int_w^x V\frac{dU}{dt}dt$$

$$\int_w^x U\,dV = UV\Big|_w^x - \int_w^x V\,dU$$

MORE ON REMAINDER

$$f(x) = f(0) + xf'(0) + \ldots + \frac{x^n}{n!}f^n(0) + R_n(x,0)$$

$$R_n = \frac{1}{n!}\int_0^x (x-t)^n f^{n+1}(t)\,dt$$

EX) $f(x) = \ln(x+1)$ $\quad f(0) = 0$

$f'(x) = 1/x+1$ $\quad f'(0) = 1$

$f''(x) = -1/(x+1)^2$ $\quad f''(0) = -1$

$f'''(x) = 2/(1+x)^3$ $\quad f'''(0) = 2$

$\vdots$

$f^n(x) = (n-1)!/(1+x)^n$

$\ln(x+1) \cong x$

$\ln(x+1) = x + R_2$

$$R_2(x,0) = \frac{1}{2}\int_0^x (x-t)^2 f'''(t)\,dt$$

$$= \int_0^x \frac{(x-t)^2}{(t+1)^3}\,dt$$

(OVER)

$\therefore R_2(x,0) < \int_0^x (x-t)^2\,dt = \frac{x^3}{3}$

FOR: $-\frac{1}{3}(x-1)^3 \Big|_0^x = \frac{1}{3}x^3$

$R < \frac{1}{100}|x|$

$\frac{1}{2}x^2 < \frac{1}{100}|x|$

$|x| < \frac{1}{50}$

1-15-70

$f(x) = \cancel{\sum_{k=0}^{n}} \Sigma a_k x^k$

$R_n(x,0)$

$f(x) = \sum_{k=0}^{n} a_k (x-a)^k$

INTERESTED IN $|R_n(x,a)|$

$R_n(x,a) = \frac{1}{n!}\int_a^x (x-t)^n f^{n+1}(t)\,dt$

ESTIMATE FROM PUTTING

$|f^{n+1}(t)| < M \Rightarrow$

$R_n(x,a) < \frac{M}{n!}\int_a^x (x-t)^n\,dt$

$$R_n = f^{(n+1)}(c)\,\frac{(x-a)^{n+1}}{(n+1)!} \qquad a<c<x$$

EX) $\sin x = x - x^3/6$

$$R_3 = 1/6 \int_0^x (x-t)^3 \sin t \, dt$$

$$R_4 = \frac{1}{24} \int_0^x (x-t)^3 \cos t \, dt$$

EX) $\sqrt{1+x} = 1 + \frac{1}{2}x$

$$|R_1| = \frac{1}{4}\int_0^x \left[(x-t)/(1+t)^{3/2}\right] dt \qquad f = \sqrt{1+x}$$

$$f' = \frac{1}{2}\frac{1}{\sqrt{1+x}}$$

$$f'' = -\frac{1}{4}\frac{1}{(1+x)^{3/2}}$$

$1+x>1$

$\frac{1}{(1+x)^{3/2}} < 1$

$$R_1 \leq \frac{1}{4}\int_0^x (x-t)\,dt = \frac{1}{8}x^2$$

IF $x<0$

$$\sqrt{1-x} = 1 - \frac{1}{2}x$$

$$|R_1| = \frac{1}{4}\int_0^x (x-t)\,dt/(1-t)^{3/2}$$

$\frac{1}{(1+t)^{3/2}} < 1$

$t<x$

$1-t > 1-x$

$\frac{1}{(1-t)^{3/2}} < \frac{1}{(1-x)^{3/2}}$

$$|R_1| \leq \frac{1}{4}\frac{1}{(1-x)^{3/2}}\int_0^x (x-t)\,dt$$

$$= \frac{x^2}{8(1-x)^{3/2}}$$

ABSOLUTE CONVERGESES

A SERIES, IS SAID TO BE.

$$\sum_{n=0}^{\infty} a_n$$

ABSOLUTELY CONVERGENT

IF $\sum_{n=0}^{\infty} |a_n|$ IS CONVERG.

IF NOT ABSOLUTELY CONV.,
THEN IT IS CALLED
RELATIVELY CON.

EX) $\sum a_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \ldots$ CONV

$\sum |a_n| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \ldots$ DIV

ABSOLUTELY CONVERSIANT POWER SERIES MAY BE
ADDED, MULTIPLIED, DIFFERENTIATED, & INTEGRATING

EX) 1) $f(x) = \sum_{n=0}^{\infty} a_n x^n$

$$f'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

2) $\sin x \cos x = \left(x - \frac{x^3}{3} + \ldots\right)\left(1 - \frac{x^2}{2} \ldots\right)$

$$= \frac{1}{2}\left(2x - \frac{8x^3}{6} \ldots\right)$$

$$\ln(1+x) = \int_0^x \frac{dt}{(1+t)} = \int_0^x (1-t+t^2\ldots)\,dt$$
$$(t<1)$$

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3}\ldots$$

$$\text{ARCTAN } x = \int_0^x \frac{dt}{1-t^2}$$

648

1-16-70

7) FIND $\ln(x+1)$ TO 3 D IF

$x = .5$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

$x = .25$

$x^2/2 = .0313 -$

$.2187$

$x^3/3 = .0052 +$

$.2239$

$x^4/4 = .0010 -$

$.2229$

FOUR TERMS SEEMS ENOUGH. CHECK FOR $n=4$:

$$|R_4| \le \frac{1}{5}\,\frac{(.25)}{(.75)} = .000266 < .0005$$

*FROM FORM. (7) p 647 FOR R_n

$$R_n = \frac{1}{n+1}\left(\frac{x^{n+1}}{1-|x|}\right)$$

EX) FIND ARCTAN 1.02 TO 3 D

USE TAYLOR SERIES:

$$f(x) = f(a) + (x-a)f'(a) + (x-a)^2 \tfrac{1}{2} f''(a) \ldots$$

WITH $f(x) = \text{ARCTAN}$, $a = 1$, $f(1) = \pi/4$

$f'(x) = \dfrac{1}{1+x^2}$, $f'(1) = \dfrac{1}{2}$

$f''(x) = \dfrac{-2x}{(1+x^2)^2}$, $f''(1) = \dfrac{1}{2}$

TAKE $x = 1.02$

$f(0) = .7854$

$\frac{1}{2} \times (.02) = .0100$

$.7954$

$-\frac{1}{4}(.02^2) = .0001$ $\therefore$ NOT NESCISARY

ARCTAN $1.02 = .797$

$$R_n(x,a) = \frac{1}{n!}\int_a^x (x-t)^n f^{n+1}(t)\,dt$$

$n = 1$

~~$R_1(x,a) = \frac{1}{1}\int^x (x-t)$~~

$$|R_n(x,1)| = \left|\int_1^x (x-t) f''(t)\,dt\right| =$$

$$\left|2\int_1^x \frac{(x-t)t\,dt}{(1+t^2)^2}\right| \le 2\int_1^x (x-t)t\,dt$$

OR $|R_1(x,1)| \le \frac{1}{3}x^3 - x + 2/3$

$\therefore |R_1(1.02,1)| \le (.0004) < .0005$

$$\frac{\sin t}{t} = 1 - \frac{t^2}{3!} + \frac{t^4}{5!} \ldots \text{ ETC.}$$

FINDING $\sqrt[3]{9}$

$$\sqrt[3]{9} = 2\sqrt[3]{1+\tfrac{1}{8}}$$

$$\sqrt[3]{25}$$

$$\sqrt[3]{27-2} = 3\sqrt[3]{1-\tfrac{2}{27}}$$

$$f(x) = \sqrt[3]{1+x}$$

EX) $\lim_{x \to 0} \frac{\sin x - \tan x}{x^3}$

$$\approx \frac{(x - \frac{1}{6}x^3) - R_{5_1} - x + \frac{1}{3}x^3 + R_{5_2}}{x^3}$$

$$\approx \frac{1}{6} + (R_{5_1} + R_{5_2})x^2 \qquad \text{ETC.}$$

1-18-70

THEOROM 9:

IF $\sum |a_k|$ CONVERGES; THEN $\sum a_k$ CONVERGES

(BUT NOT THE CONVERSE)

(THEY DON'T HAVE THE SAME LIMIT)

THEOROM 10:

IF THE POWER SERIES $\sum_0^\infty a_n x^n$ CONVERGES FOR $x=c$, THEN IT CONVERGES ABSOLUTELY FOR $|x|<|c|$. IF IT DIVERGES FOR $x=d$, THEN IT DIVERGES FOR $|x|>|d|$

THEROM X

GEMOMETRIC SERIES CONVERGES IF $|r|<1$

THEOROM 11:

A STRICTY ALTERNATING SERIES CONVERGES IF

i) $\lim_{n\to\infty} a_n = 0$ ii) $a_{n+1} \le a_n$

WHERE THE SERIES IS WRITTEN $\sum_0^\infty (-1)^n a_n$.

TO TEST THE POWER SERIES $\sum_0^\infty a_n x^n$ FOR CONVERGENCE,

i) APPLY THE RATIO

TEST TO $\sum_0^\infty |a_n||x|^n$. IF THIS SERIES IS CONVERGENT FOR $x < R$, THEN $\sum_0^\infty a_n x^n$ CONVERGES FOR $-R < x < R$

II) TEST SEPARATELY FOR $x = R$ AND $x = -R$

EXAMPLE OF THEOROM 10:

$-\ln(-x+1) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4$

IS THIS SERIES CONVERGENT?

$x < 1$

IF A SERIES IS CONVERGENT AT $\frac{1}{2}$, IT IS CONVERGENT DOWN TO, BUT NOT INCLUDED NECISSARILY, AT $-1/2$

EX) $S = \sum_{n=1}^\infty (-1)^n \frac{x^{2n+1}}{2n+1}$ CONVERGENT? IF SO FOR WHAT VALUES OF X?

BY THEOROM 8

$S^* = \sum_{n=0}^\infty \frac{|x|^{2n+1}}{2n+1}$

USE RATIO TEST $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$

$\lim_{n\to\infty} \frac{|x|^{2n+3}}{2n+3} \cdot \frac{2n+1}{|x|^{2n+1}}$

$\lim_{n\to\infty} |x|^2 \frac{2n+1}{2n+3} = |x|^2$

$\left(\lim_{n\to\infty} \frac{2n+1}{2n+3} = \lim_{n\to\infty} \frac{2+1/n}{2+3/n} = 1 \right)$

$\therefore$ FOR CONVERGENCE, $|x|<1$

AND BECAUSE OF THEROM 10,

$\Sigma\, (-1)^n\, x^{2n+1}/2n+1$ MUST

ALSO THEN ~~IS~~ CONVERGE

FOR ALL $|x|<1$, AND

DIVERGES FOR ALL $|x|>1$

WHAT ABOUT $x=1$?

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots$$

$$\frac{1}{2n+3} < \frac{1}{2n+1},$$

$\therefore$ THE SERIES CONVERGES

AT $x=1$

WHAT ABOUT $x=(-1)$

$$-\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} \qquad (-1)^{2n+1} = -1$$

$\therefore$ THE SERIES IS MERELY

THE NEGATIVE OF $x=1$

CONCLUSION:

Σ CONVERGES FOR

$-1 \le x \le 1$

EX) $S = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x+1)^n}{n}$

$S^* = \sum_{n=1}^{\infty} \frac{(x+1)^n}{n}$

RATIO TEST

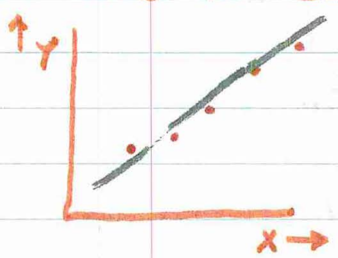
$\lim \frac{a_{n+1}}{a_n} = \lim \frac{|x+1|^{n+1}}{n+1} \cdot \frac{n}{|x+1|^n} =$

$= |x+1| \lim \left(\frac{n+1}{n+1}\right)$

$-2 < x \leq 0$

1-22-70

METHOD OF LEAST SQUARES



WHAT IS BEST LINE THRU THESE POINTS?

x	Y_{OBSER}	Y_{LINE}	$(Y_{OBS} - Y_{LINE})^2$
1	2.34	2.1	

WANT $\sum_{k=1}^{n} (Y_{OBS} - Y_{LINE})^2$ AT A MINIMUM ; $Y = mx + b$

$f(m,b) = \sum_{k=1}^{n} (Y_{OBS_k} - mx_k - b)^2$

$\therefore \frac{\partial f}{\partial m} = 0$; $\frac{\partial f}{\partial b} = 0$

$(Y_{k_{OBS}} = Y_k)$

$f(m,b) = \sum_{k=1}^{n} (Y_k^2 + m^2 x_k^2 + b^2 - 2mx_kY_k - 2bY_k + 2mbx_k)$

$= \sum_{k=1}^{n} (Y_k^2 + m^2 \Sigma x_k^2 + nb^2 - 2m \Sigma x_k Y_k - 2b \Sigma Y_k + 2mb \Sigma x_k)$

$$\frac{\partial f}{\partial m} = 2m\sum x_k^2 - 2\sum x_k y_k + 2b\sum x_k = 0$$

$$\frac{\partial f}{\partial b} = 2nb - 2\sum y_k + 2m\sum x_k = 0$$

$$m\sum x_k^2 + b\sum x_k = \sum x_k y_k$$

$$m\sum x_k + bn = \sum y_k$$

FROM THESE 2 EQUATIONS, m AND b MAY BE DETERMINED.

$$m = \frac{\begin{vmatrix} \sum y_k & n \\ \sum x_k y_k & \sum x_k \end{vmatrix}}{\begin{vmatrix} \sum x_k & n \\ \sum x_k^2 & \sum x_k \end{vmatrix}}$$

$$b = \frac{\begin{vmatrix} \sum x_k & \sum y_k \\ \sum x_k^2 & \sum x_k y_k \end{vmatrix}}{\begin{vmatrix} \sum x_k & n \\ \sum x_k^2 & \sum x_k \end{vmatrix}}$$

EXAMPLE OF APPLICATION:

$V = E - Ir$

ADD UP MEASURED VOLTAGES & CURRENTS, ADD UP THEIR SQUARS, & PLUG INTO MATRICES

MAY ADD UP LEAST SQUARES
EITHER IN X OR Y DIRECTION.

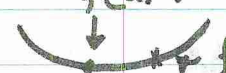
$$F(t) = f(a+ht, b+kt)$$
$$F(t) = F(0) + t\frac{dF}{dt}\Big|_{t=0} + \frac{1}{2}t^2\frac{d^2F}{dt^2}\Big|$$

1-23-70

$$F(t) = f(a+th, b+th)$$
$$= F_0 + tF'(0) + \frac{1}{2}t^2F''(0) \ldots$$
$$F'(t) = h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} = f\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)$$ COMPARE WITH $a\cdot\nabla$
$$F''(t) = f\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)$$
$$= h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x\partial y} + k^2\frac{\partial^2 f}{\partial^2 y}$$

$$F(t) = f(a,b) + t\{\underbrace{h f_x(a,b) + k f_y(a,b)}_{=0 \text{ FOR EXTREME VALUE}}\} + \frac{1}{2}t^2\{h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)\} + \ldots$$

f(a,b)



f(a+th, b+th)

FOR EXTREME VALUE, SET SECOND DIRIVITIVE IN SERIES TO 0.

MINIMUM IF $f(a+th, bt+k) - f(a,b) > 0$

LET $\phi = \frac{1}{f_{xx}}\left[h^2 f_{xx}^2 + 2hk f_{xx} f_{xy} + k^2 f_{xx} f_{yy}\right]$
$$= \frac{1}{f_{xx}}\left[(h f_{xx} + k f_{xy})^2 + k^2(f_{xx}f_{yy} - f_{xy}^2)\right]$$

MAXIMUM $f_{xx} < 0,\ f_{xx}f_{yy} - f_{xy}^2 > 0$

MINIMUM $f_{xx} > 0,\ f_{xx}f_{yy} - f_{xy}^2 > 0$

SADDLE PT. $f_{xx}f_{yy} - f_{xy}^2 < 0$

DIFFERENTIAL EQUATIONS

D METHOD DOESN'T ALWAYS WORK

TO HOT

$Y' + aY = X^2$

$(D+a)Y = X^2$

$Y = \frac{X^2}{D+a}$

$= \frac{1}{a}\left(1 - \frac{D}{a} + \frac{D^2}{a^2} - \frac{D^3}{a^3} + \dots\right)X^2$

LAPLACE TRANSFORM

EXTENSION OF D METHOD

EXAMPLE: RADIO ACTIVE DECAY

DAUGHTERS FROM PARENT

$\Delta N_d = (\lambda_p N_p - \lambda_d N_d)\Delta t$

$\frac{dN_d}{dt} + \lambda_d N_d = \lambda_p N_p$

$\frac{dN_1}{dt} + \lambda_1 N_1 = 0$

$\underline{N_1 = N_0 e^{-\lambda_1 t}}$

$\frac{dN_2}{dt} + \lambda_2 N_2 = \lambda_1 N_0 e^{-\lambda_1 t}$

MUST INTRODUCE AN INTEGRATING FACTOR $= e^{\lambda_2 t}$

$e^{\lambda_2 t}\left(\frac{dN_2}{dt} + \lambda_2 N_2\right) = \frac{d}{dt}\left(e^{\lambda_2 t} N_2\right)$

$= \lambda_1 N_0 e^{(\lambda_0 - \lambda_1)t}$

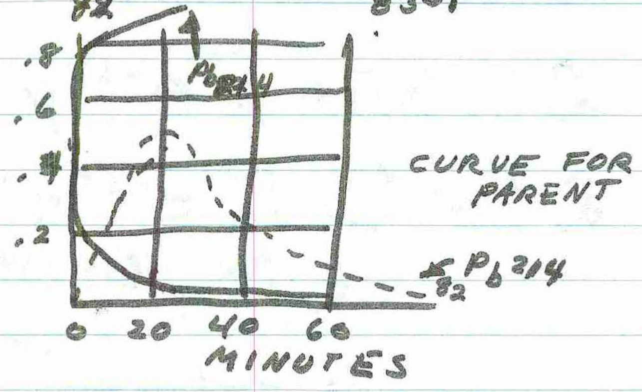
$e^{\lambda_2 t} N_2 = \frac{\lambda_1 N_2}{\lambda_2 - \lambda_1} e^{(\lambda_2 - \lambda_1)t} + C$

AT $T=0$; $N_2 = 0 \Rightarrow C = 0$

$N_2 = \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1}\left[e^{-\lambda_1 t} - e^{-\lambda_2 t}\right]$

DECAY CONSTANT $= \frac{\ln 2}{T}$

$_{84}Po^{2.8} \xrightarrow{3.05\text{ MIN}} {}_{82}Pb^{214} \xrightarrow{26.8\text{ MIN}} {}_{83}Bi^{214} \xrightarrow{19.7\text{ MIN}} Po^{214}$



1-26-70

$$\int_0^\infty e^{-Dt}\frac{dY}{dt}dt = \int_0^\infty e^{-Dt}dY$$

$$= e^{-Dt}y\Big|_0^\infty + D\int_0^\infty e^{-dt}y\,dt$$

$$= -y_0 + D\int_0^\infty e^{-Dt}y\,dt$$

$$\int_0^\infty \frac{dY}{dt}e^{-Dt}dt + a\int_0^\infty e^{-Dt}y\,dt = 0$$

$$-y_0 + (D+a)\int_0^\infty e^{-Dt}y\,dt = 0$$

$$\int_0^\infty e^{-Dt}y\,dt = \frac{Y_0}{D+a}$$

$$(D+a)Y = e^{-Ut}$$

$$-Y_0 + (D+Y)\int_0^\infty e^{-tD}y(t)\,dt$$

$$= \int_0^\infty e^{-Ut}e^{-Dt}dt$$

LAPLACE TRANSFORM

$$\mathcal{L}[Y(t)] = \int_0^\infty e^{-st} y(t)\,dt = \bar{Y}(s)$$

$$\mathcal{L}[e^{-ut}] = \int_0^\infty e^{-st} e^{-ut}\,dt$$

$$= \int_0^\infty e^{-(s+u)t}\,dt$$

$$= -\frac{1}{s+u} e^{-(s+u)t} \Big|_0^\infty$$

$$= 1/s+u$$

$$\therefore \mathcal{L}[e^{-ut}] = 1/s+a$$

LET $\int_0^\infty e^{-st} Y(t)\,dt = \bar{Y}(s)$

WE HAVE FOUND THAT:

$$\int_0^\infty e^{-st} \frac{dy}{dt}\,dt = -Y(0) + s\int_0^\infty e^{-st} Y(t)\,dt$$

$$= -Y_0 + s\bar{Y}(s)$$

$$-Y(0) + s\bar{Y}(s) = Y(0)$$

$$\bar{Y}(s) = Y(0)/s+\alpha$$

$$Y(t) = e^{-\alpha t} Y(0)$$

$$Y(f) \xrightarrow{\text{CALC}} \bar{Y}(s) \xrightarrow{\text{ALGEBRA}} F(s) \xrightarrow{\text{CALC}} F(t)$$

LINEAR TRANSFORMATION

$$Y(t) = \lambda Y_1(t) + \mu Y_2(t)$$

$$\mathcal{L}[Y] = \int_0^\infty e^{-st}(\lambda Y_1 + \mu Y_2)\,dt$$

$$= \lambda \int_0^\infty e^{-st} Y_1(t)\,dt + \mu \int_0^\infty e^{-st} Y_2(t)\,dt$$

$$= \lambda \mathcal{L}[Y_1] + \mu_0 \mathcal{L}[Y_2]$$

$$\mathcal{L}[\cosh at] = \mathcal{L}\left[\tfrac{1}{2}e^{at} + \tfrac{1}{2}e^{-at}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{s}{s^2 + a^2}$$

$$\mathcal{L}[t^n] = \int_0^\infty e^{-st} t^n\,dt$$

1-27-70

$f(t)$	$\int_0^\infty e^{-st} f(t)\,dt = \mathcal{L}(f(t))$	
e^{at}	$\frac{1}{s-a}$	$s > a$
$\cosh at \Rightarrow$	$s/s^2 - a^2$	$s > a$
$\sinh at \Rightarrow$	$a/(s^2 - a^2)$	$s > a$
$t^n \Rightarrow$	$n!/(s^{n+1})$	$s > 0$
$\cos \omega t \Rightarrow$	$s/(s^2 + \omega^2)$	$s > 0$
$\sin at \Rightarrow$	$\omega/(s^2 + \omega^2)$	$s > 0$

$e^{i\omega t} = \cos\omega t + i\sin\omega t$

$$\int_0^\infty e^{-st} e^{i\omega t}\,dt = \int_0^\infty e^{-(s-i\omega)t}\,dt$$

$$= \frac{-1}{(s-i\omega)} e^{-(s-i\omega)t}\Big|_0^\infty$$

$$= -\frac{e^{-st}(\cos\omega t + i\sin\omega t)}{s-i\omega}\Big|_0^\infty$$

$$= \frac{1}{s-i\omega}$$

$$= \frac{s+i\omega}{s^2+\omega^2}$$

f(t) FOR t<0 IS IRRELEVENT
FOR SIMPLICITY ASSUME
f(t) = 0 FOR t<0

$\mathcal{L}\{f(t)\}$ $\quad f(t) = \begin{cases} t & 0<t<4 \\ 5 & t>4 \end{cases}$

$$\int_0^4 e^{-st}t\,dt + 5\int_4^\infty e^{-st}\,dt$$

$$-\frac{1}{s}te^{-st}\Big|_0^4 + \frac{1}{s}\int_0^4 e^{-st}\,dt + 5\int_4^\infty e^{-st}\,dt$$

$-\frac{4}{s}e^{-4s} - \frac{1}{s^2}e^{-st}\Big|_0^4 - \frac{5}{s}e^{-st}\Big|_4^\infty$

$-\frac{4}{s}e^{-4s} - \frac{e^{-4s}}{s^2} + \frac{1}{s^2} + \frac{5}{s}e^{-4s}$

$\frac{1}{s^2} + \left(\frac{1}{s} - \frac{1}{s^2}\right)e^{-4s}$

HEAVISIDE FUNCTION:

$H(t-a) = \alpha(t-a) = \begin{cases} 0 & t<a \\ 1 & t>a \end{cases}$

$\int_0^\infty e^{-st} H(t-a)\,dt = \int_a^\infty e^{-st}\,dt$

$= -\frac{1}{s}e^{-st}\Big|_a^\infty$

$= e^{-as}/s$

1-29-70

3) $\mathcal{L}\{\frac{1}{2}t^3 + t^2 - 1\} = \frac{1}{2}\frac{6}{s^4} + \frac{2}{s^3} - \frac{1}{s}$ $\quad = 3!$

$\mathcal{L}(\sin t) = 1/(s^2+1)$

$\mathcal{L}(e^{at}\sin t) = 1/((s-a)^2+1)$

$\mathcal{L}\{e^{at}F(t)\} = \int_0^\infty e^{-st}e^{at}F(t)\,dt$

$= \int_0^\infty e^{-(s-a)t}F(t)\,dt$

(COMPARE WITH $\mathcal{L}(F(t)) = \int_0^\infty e^{-st}F(t)dt$)

LET $\bar{Y} = \mathcal{L}(Y)$

THEN $\mathcal{L}\{dY/dt\} = s\bar{Y} - Y(0)$

AND $\mathcal{L}(d^2Y/dt^2) = s^2Y' - sY(0) - Y(0)$

PROOF:

$$\mathcal{L}\{Y'\} = \int_0^\infty e^{-st} Y'(t)\,dt$$

$$= \int_0^\infty e^{-st} dY$$

$$= e^{-st} Y\Big|_0^\infty + s\int_0^\infty e^{-st} Y\,dt$$

$$= -Y(0) + s\mathcal{L}(Y)$$

$$\mathcal{L}\{Y''\} = \int_0^\infty e^{-st} Y''\,dt$$

$$= \int_0^\infty e^{-st} dY'$$

$$= e^{-st} Y'\Big|_0^\infty + s\int_0^\infty e^{-st} Y'\,dt$$

$$= -Y'(0) - sY_0 + s^2\mathcal{L}\{Y\}$$

EX) $Y'' + Y = 1$

$(~~LET~~\ \mathcal{L}(Y) = \bar{Y})$

$(\mathcal{L}(Y'') = s^2\bar{Y} - sY(0) - Y'(0))$

$$s^2\bar{Y} - sY(0) - Y'(0) + \bar{Y} = \frac{1}{s}$$

$[\mathcal{L}(1) = 1/s]$

$Y(0) = 2;\ Y'(0) = 0$

$$s^2\bar{Y} + \bar{Y} = \frac{1}{s} + 2s + 0$$

$$(s^2+1)\bar{Y} = \frac{1}{s} + 2s$$

$$\bar{Y} = \frac{1}{s(s^2+1)} + \frac{2s}{s^2+1}$$

LOOK INTO TABLE FOR RECOGNITION. MUST SPLIT UP UNFAMILIAR ONES INTO PARTIAL FRACTIONS.

$$\bar{Y} = \frac{A}{s} + \frac{B+Cs}{s^2+1} + \frac{2s}{s^2+1}$$

$$= \frac{1}{s} - \frac{s}{s^2+1} + \frac{2s}{s^2+1}$$

$$= \frac{1}{s} + \frac{s}{s^2+1}$$

$$\therefore Y = 1 + \cos t$$

$$\frac{d}{dx} \sin x = \cos x$$

$$\mathcal{L}\{\sin t) = \frac{1}{s^2+1}$$

$$\mathcal{L}\{\cos t\} = \mathcal{L}\{\tfrac{d}{dt} \sin t\}$$

$$\Leftarrow \frac{s}{s^2+1} - \sin(0) = \frac{s}{s^2+1}$$

$$\mathcal{L}\{Y'\} = s\mathcal{L}\{Y\} - Y(0)$$

CONVERSLY:

$$\mathcal{L}(\sin t\} = -\mathcal{L}\{\tfrac{d}{dt} \cos t\}$$

$$= \frac{-s^2}{s^2+1} + \cos(0)$$

$$= \frac{-s^2}{s^2+1} + 1 = \frac{1}{s^2+1}$$

1-30-70

$$\bar{Y} = \frac{1}{(s-3)(s-2)(s-1)}$$

$$= \frac{A}{s-3} + B/_{s-2} + \frac{C}{s-1}$$

$$1 = A(s-2)(s-1) + B(s-3)(s-1) + C(s-3)(s-2)$$

$$= (A+B+C)s^2 - (3A+4B+5C)s + (2A+3B+6C)$$

$$\therefore \quad A+B+C=0$$

$$3A+4B+5C=0$$

$$2A+3B+6C=1$$

OR MAY SET ~~A, B, C~~S TO CONVENIENT CONSTANTS ~~IE~~ SINCE RELATIONSHIP IS AN IDENTITY

$$s=3 \Rightarrow 1 = 2A \Rightarrow A = \tfrac{1}{2}$$

$$s=2 \Rightarrow 1 = -B \Rightarrow B = -1$$

$$s=1 \Rightarrow 1 = 2C \Rightarrow C = 1/2$$

EX 157 #7)

$y'' - 3y' + 2y = e^{3t} \quad y(0)=0;\ y'(0)=0$

$\cancel{(s^2\bar{y} - s) - 3(s\bar{y}}$

$s^2\bar{y} - s\bar{y} + 2\bar{y} = \frac{1}{s-3}$

$(s-1)(s-2)\bar{y} = \frac{1}{s-3}$

$\bar{y} = \frac{1}{(s-1)(s-2)(s-3)}$

FROM PREEEEEDING PAGE:

$y = \frac{1}{2}e^{3t} - e^{2t} + \frac{1}{2}e^{t}$

$f(s) = 1 \Rightarrow$ NO $f(t) \Rightarrow$ NO $\mathcal{L}$

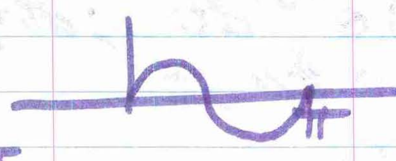
FOR $\int_0^\infty e^{-st} f(t)\,dt$ DOESN'T

EXIST TO WELL.

EX 154 #16)

$B(t) = \sin 2t \quad 0 < t < \pi$

$= 0 \quad t > \pi$



$\int_0^\pi e^{-st} \sin 2t\,dt$

USE $e^{i\alpha} = \cos\alpha + i\sin\alpha$

$$f(s) = \int_0^\infty e^{-st} F(t)\,dt$$

$$\frac{d f(s)}{ds} = \int_0^\infty -t e^{-st} F(t)\,dt$$

$$= \int_0^\infty e^{-st}(-t F(t))\,dt$$

$$\therefore \mathcal{L}\{-t F(t)\} = f'(s)$$

Ex)

$$\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}$$

$$\mathcal{L}\{-t \sin kt\} = \frac{d}{ds} \frac{k}{s^2+k^2}$$

$$= \frac{-2ks}{(s^2+k^2)^2}$$

INVERSE TRANSFORMS:

$$\int_0^\infty F(at)\, e^{-st}\,dt \quad ; at = \mu$$

$$\frac{1}{a} \int_0^\infty F(\mu)\, e^{-\frac{s}{a}\mu}\,dt$$

$$\frac{1}{a} f\left(\frac{s}{a}\right)$$

$$\mathcal{L}\{F(T/n)\} = f(bs)$$

EX) $\frac{1}{s-3} = \mathcal{L}\, e^{3t}$

$\frac{1}{2s-3} = \mathcal{L}\tfrac{1}{2}\, e^{\frac{3}{2}t}$

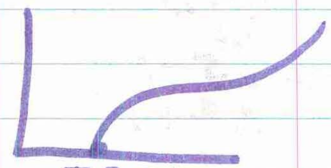
$\mathcal{L}\{F(t-c)\alpha(t-c)\}$

$= e^{-cs} f(s) \quad ; \alpha(t-c) = \begin{cases} 0 & t<c \\ 1 & t>c \end{cases}$

$(f(s)) = \mathcal{L}\{f(t)\}$

PROOF:

$\int_c^\infty F(t-c)\alpha(t-c)e^{-st}\,dt$



LET $t-c=u$

$\int_0^\infty F(u)\, e^{-(u+c)s}\,dt$

$= e^{-sc}\int_0^\infty e^{su} F(u)\,du$

$= e^{-sc} f(s)$

2-2-70

13) $\dfrac{s^2+2s+5}{(s^2+2s+2)(2s^2+2s+5)}$

$\Rightarrow \frac{1}{2}e^{-t}(\sin t + \sin 2t)$

14) $\dfrac{s^2}{(s^2+4)^2} \Rightarrow \frac{1}{2}t\cos 2t + \frac{1}{4}\sin 2t$

15) $\dfrac{s+1}{(s^2+2s+2)^2} \Rightarrow \frac{1}{2}te^{-t}\sin t$

EX) $\dfrac{s}{s^2+2s+2} = \dfrac{s}{(s+1)^2+1}$

$\Downarrow$

$\dfrac{s}{s^2+1} - \dfrac{1}{s^2+1} \Leftarrow \dfrac{s-1}{s^2+1}$

$= \mathcal{L}\{\cos t - \sin t)$

$\Rightarrow \mathcal{L}^{-1}\left\{\dfrac{s}{(s+1)^2+1}\right\} = e^{-t}(\cos t - \sin t)$

EX) $\dfrac{3e^{-s}}{(s-1)(s+2)} = e^{-s}\left[\dfrac{3}{(s-1)(s+2)}\right]$

$= e^{-s}\left[\dfrac{1}{s-1} - \dfrac{1}{s+2}\right]$

CONT.

CAN USE PARTIAL FRACTIONS ONY WHEN THERE IS A POLYNOMIAL IN BOTH NUMERATOR AND DENOMINATOR

$\mathcal{L}^{-1}\left\{\frac{1}{s-1} - \frac{1}{s+2}\right\} = e^{t} - e^{-2t}$

THEN

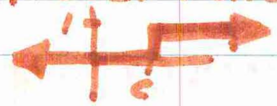
WHAT IS $\mathcal{L}^{-1}\left\{e^{-s}\left(\frac{1}{s-1} - \frac{1}{s+2}\right)\right\}$?

$$= \left[e^{t-1} - e^{-2(t-1)}\right] \alpha(t-1)$$

α = UNIT STEP FUNCTION.

FOR $t<1$, $\alpha=0$; $t>1$, $\alpha=1$

UNIT STEP FUNCTION α



$$\alpha(t-c) = \begin{matrix} 1 & t>c \\ 0 & t<c \end{matrix}$$

EX) $\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^2+4s+8}\right\}$ ← a

$\frac{1}{s^2+4s+8} = \frac{1}{(s+2)^2+4}$ ← b

$\downarrow$

$\frac{1}{s^2+4} \rightarrow \mathcal{L}\left\{\frac{1}{2}\sin 2t\right\}$

WHAT IS TRANSFORM OF b ?

$\mathcal{L}\left\{\frac{1}{2} e^{-2t} \sin 2t\right\}$

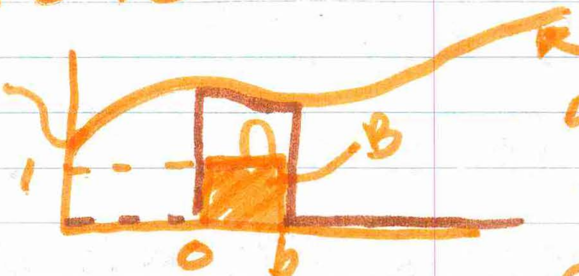
WHAT IS THE VALUE OF a ?

MUST USE EXPONENTIAL SHIFT

$\frac{1}{2} e^{-2(t-3)} \sin 2(t-3)$

OR $\frac{1}{2} e^{-2(t-3)} \sin 2(t-3) \quad \alpha(t-3)$

2-6-70

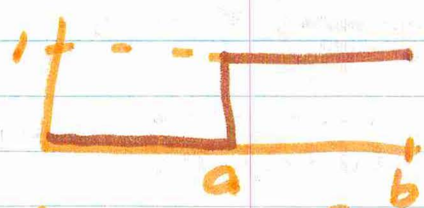


F(t) DEFINED FOR $t > 0$

$\therefore$ FUNCTION IN ORANGE ~~BROWN~~ TIMES
F(t) = F IN BROWN.
OARNGNE FUNCTION = B

$$B = \alpha(t-a) - \alpha(t-a)$$

WHERE α =



$$\therefore \alpha(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \\ \frac{1}{2} & t = 0 \end{cases}$$

IF $a < b$

$$\Rightarrow B = \begin{cases} 0 & t < a \\ 1 & a < t < B \\ 0 & t > b \end{cases}$$

$\therefore$ F IN BROWN =

$$F(t)\left[\alpha(t-a) - \alpha(t-b)\right]$$

TAKE $X^2Y'' + XY' - Y = 0$

$Y = a_0 + a_1 X + a_2 X^2 \ldots$

$Y' = \quad a_1 + 2a_2 X \ldots$

$Y'' = \quad 2a_2 + 6a_3 X \ldots$

$2a_2X^2 + 6a_3X^3 + a_1X + 2a_2X^2 + 3a_3X^3$

$- a_0 - a_1X - a_1X^2 - a_3X^3 = 0$

$-a_0 + 3a_2X^2 + 8a_3X_3 = 0$

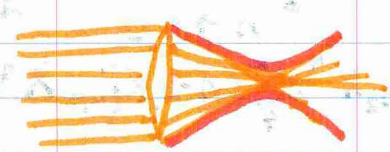
1ST SOL $\rightarrow Y_1 = a_1 X$

2nd SOL $\rightarrow Y_2 = 1/X$

2-23-70

AIRY EQUATION $\quad Y'' = XY$

LIGHT DIFFRACTION NEAR A CAUSIC SURFACE



SUPPOSE THAT:

$Y = \sum_0^\infty a_n X^n$ IS A SOLUTION

$$Y'' = \sum_{n=0}^{\infty} n(n-1) a_n X^{n-2}$$

$\longrightarrow$

$$\overset{y''}{\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}} = \overset{xy}{x \sum a_n x^n} = \sum_{n=0}^{\infty} a_n x^{n+1}$$

$$2a_2 + 6a_3 x^2 + \overset{+12a_4 x^2}{20x^3} + 30a_6 x^4 + \ldots$$
$$= a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4 + \ldots$$

$a_2 = 0$ OR $a_4 = \frac{1}{120},$

$6a_3 = a_0$ $a_6 = \frac{1}{30} a_3$

$12a_4 = a_1$ $a_7 = \frac{1}{42} a_4 = \frac{1}{2\cdot 4\cdot 5\cdot 7} a_1$

$20a_5 = a_2 = 0$ $a_3 = \frac{1}{6} a_1$

$20a_6 = a_3$ $a_8 = \frac{1}{56} a_5 = 0$

$42a_7 = a_4$

$\vdots \quad \vdots$

$a_4, a_7, a_{10}, a_{13} \ldots a_1$

$a_3, a_6, a_9, a_{12} \ldots a_0$

$a_2 = a_5 = a_8 = 0$

2-26-70

$$\sum_{0\,(2)}^{\infty} n(n-1) C_n x^{n-2} + \sum_{0}^{\infty} n C_n x^{n+3} - 3 \sum_{2}^{\infty} C_n x^n = 0$$

$n \to n+2 \qquad n \to n-3$

$$\sum_{n=2}^{\infty} (n+2)(n+1) C_{n+2} x^n + \sum_{4}^{\infty} (n-3) C_{n-3} x^n - 3 \sum_{n=2}^{\infty} C_n x^n = 0$$

$$\sum_{n=4}^{\infty} \left[(n+2)(n+2) C_{n+2} + (n-3) C_{n-3}\, C_n \right] x^n$$

$\Big(2C_2 + 6C_3X + 12C_4X^2\Big)$ → LEFT OVER FROM FIRST SERIES

$-12C_4X^2 - 3C_2X^2) = 0$

n=3 $20C_5X^3 - 3C_3X^3$

$C_2 = 0$

$C_3 = 0$

$C_2 = 4C_4 \Rightarrow C_4 = 0$

$$C_{n+2} = \frac{3C - (n-3)C_{n-3}}{(n+2)(n+1)}$$

3·2·70

SOMETIMES SERIES METHOD AIN'T SO HOT

$\underset{P}{8X^2}Y'' + \underset{Q}{10X}Y' - \underset{R}{(1+X)}Y = 0$

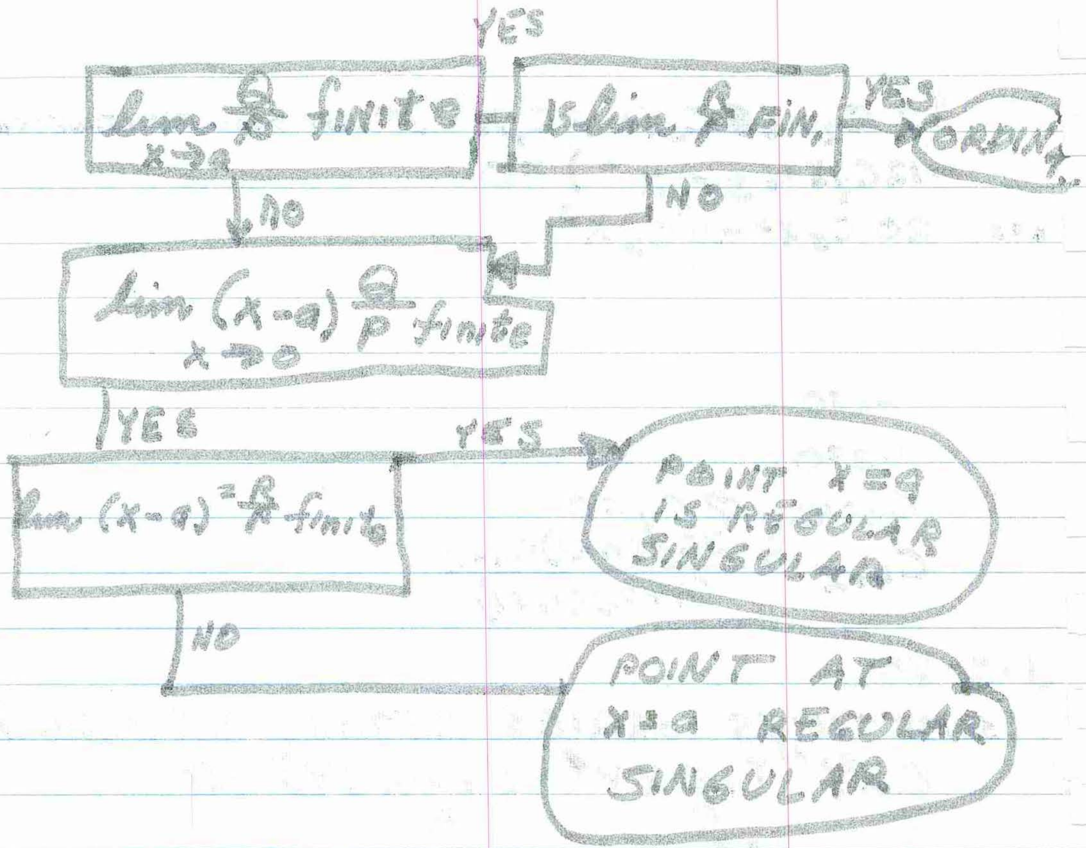
$\frac{Q}{P} = \frac{10X}{8X^2}$ $\lim_{X\to 0} \frac{Q}{P} = \infty \Rightarrow X = 0$, SING.

A) IS $\lim_{X\to a} \frac{Q}{P}$ FINITE) (IS $\lim_{X\to a} \frac{R}{P}$ IS FINITE)

1) IF BOTH YES, POIN IS EXTRA ORDINARY

COVER

$Py'' + Qy' + Ry = 0$



Ex) $8x^2y'' + 10xy' - (1+x)y = 0$
(REGULAR SINGLE POINT)

$$y = x^s \sum_{n=0}^{\infty} a_n x^n$$

3-5-70

$$a_n = \frac{(\ )(\ )}{(\ )(\ )} a_{n-2}$$

IF DEGREE OF NUMERATORS ORDER SMALLER THAN DENOMINATOR

$$\lim_{n\to\infty} \frac{a_n}{a_{n-2}} \to 0$$

$\therefore$ SERIES ~~OF A~~ CONVERGES FOR ALL X

EX)

IF $a_n = \frac{n+1}{n-2} a_n$

THEN SOMETHING HAPPENS

(4-3) $2XY'' + (1+2X)Y' - 5Y = 0$

$$Y = \sum a_n X^{n+c}$$

$$\cancel{2\sum(n+c)(n+c-1)a_n X^{n+c}}$$

$$0 = 2\sum[(n+c)(n+c-1)+(1+c)]\, a_n X^{n+c-1}$$
$$+ \sum_0^\infty [2(n+c)-5]\, a_n X^{n+c}$$

$$\sum_0^\infty (n+c)(2n+2c-1)\, a_n X^{n+c-1}$$
$$+ \sum_0^\infty (2n+2c-5)\, a_n X^{n+c} = 0$$

$$n \to n+1$$

$$0 = \sum_{-1}^\infty (n+c+1)(2n+2c+1)\, a_{n+1} X^{n+c}$$
$$+ \sum_0^\infty (2n+2c-5)\, a_n X^{n+c}$$

$$c(2c-1) = 0 \to \text{IND EQ.}$$

$$a_{n+1} = \frac{2n+2c-5}{-(n+c+1)(2n+2c+1)} a_n \to \text{REC. REL.}$$

CONT.

(INTERESTING NO?)

FOR
$C=0$

$$a_{n+1} = -\frac{2n-5}{(n+1)(2n+1)} a_n$$

$$\text{FOR } C=\tfrac{1}{2} \Rightarrow a_{n+1} = \frac{-(n-2)\,a_n}{(n+\tfrac{3}{2})(n+1)}$$

MAY EVALUATE a_0 & a_1 $(n-2)$
ALL ELSE IS $=0$ ← INTEGER!

ANOTHER SUPER EXAMPLE
BESSEL EQUATION

$$y'' + \frac{1}{x}y' + \left(1 + \frac{n^2}{x^2}\right)y = 0$$

OR) $x^2 y'' + x y' + (x^2 - n^2) y = 0$

CONSIDER $n=0$:

$$x^2 y'' + x y' + x^2 y = 0$$

$$x y'' + y' + x y = 0$$

$x=0$ IS A R.S.P.

$$y = \sum a_n x^{n+c}$$

$$0 = \sum_{n=0}^{\infty} [(n+c)(n+c-1) + (n+c)]\, a_n x^{n+c-1} + \sum_0^{\infty} a_n x^{n+c+1}$$

$$\sum_0^{\infty} (n+c)^2 a_n x^{n+c-1} + \sum_0^{\infty} a_n x^{n+c+1} = 0$$

$n \to n-2$

$$\sum_0^{\infty} (n+c)^2 a_n x^{n+c-1} + \sum_2^{\infty} a_{n-2} x^{n+c-1} = 0$$

$$\therefore c^2 a_n = 0 \Rightarrow c = 0$$

$$(c+1)^2 a_1 = 0 \Rightarrow a_1 = 0$$

$$a_n = \frac{-a_{n-2}}{(c+n)^2}$$

$c = 0$

$$a_{2k} = \frac{-a_{2k-2}}{4k^2}$$

$$= \frac{(-1)^k a_0}{4^k (k!)^2}$$

$$\therefore y = \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k (k!)^2} \left(\frac{x^2}{4}\right)^k = a_0(x)$$

$$= \sum_{n=0}^{\infty} a_n x^{n+c} \qquad \boxed{a_0 \neq 0}$$

3-9-70

$$y'' + \frac{1}{x} y' + \left(1 - \frac{n^2}{x^2}\right) y = 0$$

or $x^2 y'' + x y' + (x^2 - n^2) y = 0$

BESSLER EQUATIONS

$$y = \sum a_m x^{m+c}$$

$$0 = \sum a_m (n+c)(n+c-1) x^{m+c} + \sum a_m (n+c) x^{m+c} + \sum a_m x^{m+c+2} - \sum a_m x^{m+c} n^2$$

$$0 = \sum_0 a(m+c+n)(m+c-n) x^{m+c} + \sum_2 a_{m-2} x^{m+c}$$

AT $m = 0$

$(n+c)(-n+c) = 0$ ETC.

$c = n$

$$a_m = -\frac{a_{m-2}}{(c+m+n)(c+m-n)}$$

$$a_{2k} = -\frac{a_{2k-2}}{4(k+n)k} = \frac{(-1)^k a_0}{k!(k+n)!\,4^k}$$

CONT

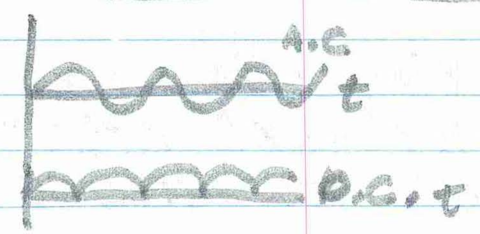
FOR $c = -n$

$$a_m = -\frac{a_{m-2}}{m(m-2n)}$$

$$a_k = -\frac{a_{2k-2}}{4k(k-n)}$$

IF $c_1 - c_2 =$ INTEGER, THEN TROUBLE

THE FOURIER SERIES



$$F(t) = A + \sum a_n \cos n\omega t + \sum b_n \sin n\omega t$$

GIVEN

$$F(x) = F(x + 2c)$$

$$\cancel{F(x) = \frac{1}{2} a_0 + \sum_{n=0}^{\infty} a_n \cos \frac{n\pi x}{c}}$$

$$\cos \frac{n\pi(x+2c)}{c} = \cos\left(\frac{n\pi x}{c} + 2\pi\right) = \cos\left(\frac{n\pi x}{c}\right)$$

$$\boxed{F(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{c}}$$

$$\int_{-c}^{c} \cos \frac{m\pi x}{c} dx = 0$$

ASSUME:

$$\int_{c}^{-c} \sin \frac{n\pi x}{c} \sin \frac{m\pi x}{c} dx = 0 \rightarrow (m \neq n) \quad = c \quad m = n \neq 0$$

$$\int_{-c}^{c} \sin \frac{n\pi x}{c} \cos \frac{m\pi x}{c} dx = 0$$

$$\int_{-c}^{c} \cos \frac{n\pi x}{c} \cos \frac{m\pi x}{c} dx = 0 \quad (m \neq n) \quad = 2c \quad m = 0$$

$$\int_{-c}^{c} F(x) \cos \frac{m\pi x}{c} dx = \frac{1}{2} a_0 \cancel{\int_{-c}^{c} \cos \frac{m\pi x}{c} dx}$$

OKAY?

$$+ \sum_{n=1}^{\infty} a_n \int_{-c}^{c} \cos \frac{m\pi x}{c} \cos \frac{n\pi x}{c} dx$$

$$+ \sum_{n} b_n \int_{-c}^{c} \cancel{\cos \frac{m\pi x}{c}} \sin \frac{m\pi x}{c} dx$$

$$\int_{-c}^{c} F(x) dx = c a_n$$

$$\therefore a_n = \frac{1}{c} \int_{-c}^{c} F(x) \cos \frac{m\pi x}{c} dx$$

$$\int_{-c}^{c} F(x) dx = \frac{1}{2} a_0 \int_{-c}^{c} dx$$

$$+ \sum a_n \int_{-c}^{c} \cos \frac{n\pi x}{c} dx$$

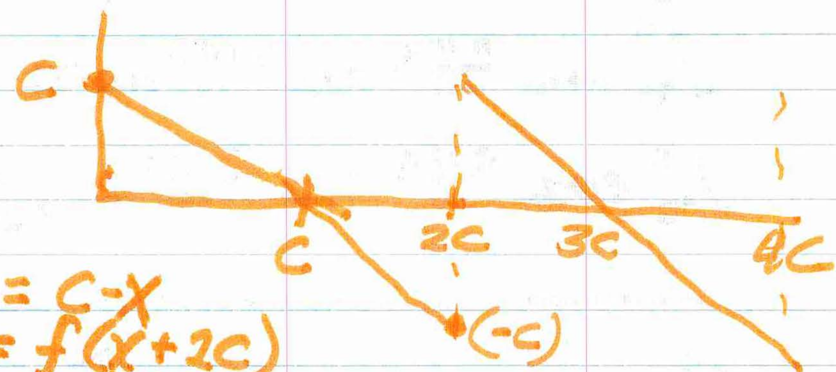
$$+ \sum b_n \int_{-c}^{c} \sin \frac{n\pi x}{c} dx$$

$$= c a_0$$

$$\therefore a_0 = \frac{1}{c} \int_{-c}^{c} F(x) dx$$

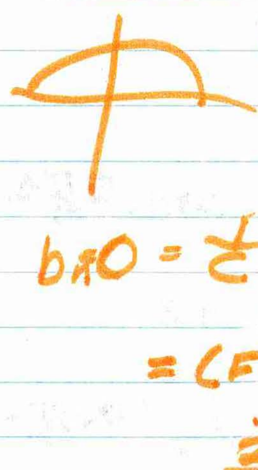
$$b_n = \frac{1}{c} \int_{-c}^{c} F(x) \sin \frac{n\pi x}{c} x \, dx$$

3-13-70
Pg 398
5)



$f(x) = c - x$
$= f(x+2c)$
$= -f(-x)$

ODD FUNCTION MEANS SYMMETRY WITH RESPECT TO ORIGIN: EVEN WITH Y AXIS.



EVEN FUNCTION USE COS SERIES

$b_n = 0 = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}\,dx$

= (FOR EVEN FUNCTIONS:

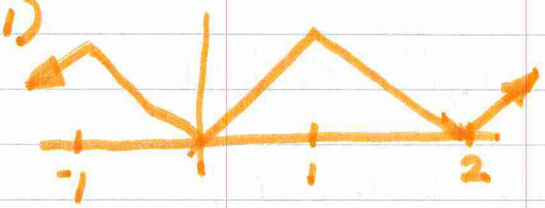
$\frac{2}{c}\int_0^c f(x)\cos\frac{n\pi x}{c}\,dx$

ODD $a_n = 0$

$b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}\,dx$

$= \frac{2}{c}\int_0^c f(x)\sin\frac{n\pi x}{c}\,dx$

FOR ODD $\Rightarrow$ Fourier SIN SERIES
FOR EVEN $\Rightarrow$ " COS "
Pg 398
1)



T FROM -1 TO $+1 = 2$

$$a_n = \int_{-1}^{1} f(x) \cos \frac{n\pi x}{1}\, dx$$

$$a_n = \int_{-1}^{0} (-x) \cos \frac{n\pi x}{1} dx + \int_0^1 x \cos \frac{n\pi x}{dx} dx$$

$x \to -s$

$$-\int_{+1}^{0} s \cos n\pi s\, ds$$

$$= \int_0^1 s \cos n\pi s\, ds$$

PARSEVAL'S THEOROM

IF $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos_n \cos \frac{n\pi x}{s} + b_n \sin \frac{n\pi x}{s})$

$$\int_{-T}^{T} [f(x)]^2 dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

CONT $\rightarrow$

$$\cancel{\pi}[f(x)]^2 = \tfrac{1}{2}a_0 f(x) + \sum_{n=1}^{\infty}(a_n f(x)\cos nx + b_n f(x)\sin nx)$$

$$\int_{-\pi}^{\pi}[f(x)]^2 dx = \tfrac{1}{2}a_0\int_{-\pi}^{\pi} f(x)\,dx + \sum_{n=1}^{\infty}\left[a_n\int_{-\pi}^{\pi} f(x)\cos nx\,dx + b_n\int_{-\pi}^{\pi} f(x)\sin nx\,dx\right]$$

$$a_n = \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$\therefore \int_{-\pi}^{\pi}[f(x)]^2 dx = \tfrac{1}{2}a_0\,\pi a_0 + \sum_{n=1}^{\infty}(a_n\pi a_n + b_n\pi b_n)$$

3-16-70

IF $f(x) = {}^{+}f(x) \rightarrow$ EVEN

$f(-x) = {}^{-}f(x) \rightarrow$ ODD

EX: $\ln|\sin x|$

$\ln|{}^{+}\sin^{-}x| = \ln|-\sin x| = \ln|\sin x|$

EVEN

EX) xe^x IS NEITHER

IS FUNCTION PERIODIC?

EX) $f(x) = \sin 6x \Rightarrow$ YES $\Rightarrow T = \frac{\pi}{3}$

EX) $\ln|\sin x| = \ln|\sin x + \pi|$

$= \ln|-\sin x|$

$\therefore T = \pi$

TOPICS COVERED THIS COURSE:

1) ALTERNATING SERIES (NOT ON FINAL)
2) TAYLOR SERIES; AROUND ORIGIN AND OTHERS & ERROR ESTIMATE
3) INTERVAL OF CONVERGENCE:
   a) RATIO TEST INTERIOR
   b) OTHER TESTS FOR END-POINTS
4) ABSOLUTE CONVERGENCE (ABSOLUTE VALUE OF SERIES) (CONDITIONAL CONVERGENCE)
5) LEAST SQUARES
6) UNDETERMINATE FORMS ~~OF~~ BY SERIES EXPANSION
7) DIRECT & REVERSE LAPLACE
   a) $\mathcal{L}$ OF PERIODIC FUNCTIONS
   b) REWRITE A FUNCTION USING STEP FUNCTIONS & $\mathcal{L}$
   c) PARTIAL FRACTIONS
   d) CONVOLUTION
   e) D.E. - INITIAL CONDITIONS
8) SERIES SOLUTION
   a) DETERMINATION OF ORD, REG. SING., IRR. SING

b) ORD. POINT:

$Y = \sum a_n x^n$

$Y = \sum a_n (x-c)^n$

c) COLLECTION OF TERMS

d) RECURANCE RELATIONS: SPECIAL VALUES

e) DOES A SERIES TERMINATE. IF NOT, DOES IT CONVERGE? WHERE DOES IT CONVERGE?

f) REG. SINGULAR POINTS (SEE ABOVE C & d) FIND 2 FUNDIMENTAL SOLUTIONS

g) FINDING RECURRENT & INDICIAL EQUATION

h) AFTER FINDING ROOTS OF IND. EQ; USUALLY TWO SERIES ARE OBTAINED. DOES ONE OR BOTH TERMINATE. IF NOT, FIND INTERVAL OF CONVERGENCE BY RATIO TEST

9) FORIER SERIES

a) FIND F.S. GIVEN f(x)-PERIODIC

b) ODD-EVEN; FUNDIMENTAL PERIOD

c) F.S. SIN & COS SERIES

d) ODD & EVEN EXPANSIONS

e) QUESTIONS ~~GONE~~ SIMILAR TO FINDING $\sum_1^\infty \frac{1}{n^2}$

f) PARSEVAL

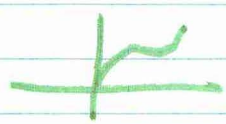
10) KNOW SERIES FOR:

$\sin x = \sum (-1)^k \frac{x^{2k+1}}{(2k+1)!}$

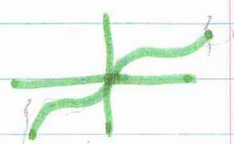
$\cos x = \sum (-1)^k x^{2k}/(2k)!$

$e^x = \sum \frac{x^n}{n!}$

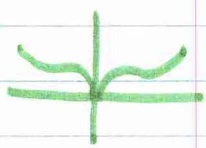
I) GIVEN



A) ODD



B) EVEN



$$\bar{f}(a) = \int_0^1 0\,dx + \int_1^2 (x-1)e^{-ax}\,dx + \int_2^\infty 0\,dx$$

$$= \left[(x-1)\left(-\frac{1}{a}e^{-ax}\right) - (1)\left(\frac{1}{a^2}e^{-ax}\right)\right]_1^2$$

$$= \frac{1}{a^2}e^{-a} - \left(\frac{1}{a}+\frac{1}{a^2}\right)e^{-2a}$$

$$\frac{2}{a}+\frac{4}{a^2} \Rightarrow g(x) = 2+4x$$

$$\frac{A}{a}+\frac{B}{a+2}+\frac{C}{a-2}$$

$$[a\,\bar{y}(a) - 0] - 2\bar{y} = \frac{6}{a^2+9} \Rightarrow \bar{y}(a) = \frac{6}{(a-2)(a^2+9)}$$

$$[a^2\bar{y}(a) - a\cdot 1 - 1] + 9\bar{y}(a) = \frac{13}{a+2} \Rightarrow (a^2+9)\bar{y}(a) = \frac{13}{a+2} + a + 1$$

So

$$\bar{y}(a) = \frac{13 + (a+1)(a+2)}{(a+2)(a^2+9)} = \frac{a^2+3a+15}{(a+2)(a^2+9)}$$

$$\frac{\partial E}{\partial b} = 0 \Rightarrow 2m\left[\int_{x_1}^{x_2} x\,dx\right] + 2b\left[\int_{x_1}^{x_2} dx\right] - 2\left[\int_{x_1}^{x_2} f(x)\,dx\right] = 0$$

$$\frac{\partial E}{\partial m} = 0 \Rightarrow 2m\left[\int_{x_1}^{x_2} x^2\,dx\right] + 2b\left[\int_{x_1}^{x_2} x\,dx\right] - 2\left[\int_{x_1}^{x_2} x f(x)\,dx\right] = 0$$

From which

$$\left[\int_{x_1}^{x_2} x\,dx\right] m + \left[\int_{x_1}^{x_2} dx\right] b = \left[\int_{x_1}^{x_2} f(x)\,dx\right]$$

[illegible]

See text or class notes

[illegible]
any constants c_1 & c_2 [illegible]

See text or class notes.

[illegible]

[illegible] Why?

The formula $\mathcal{L}\{y'(t)\} = s\mathcal{L}\{y(t)\} - y(0)$ requires that <u>both</u> $\mathcal{L}\{y(t)\}$ <u>and</u> $y(0)$ exist. But $y(0) = \left.\frac{1}{t}\right|_{t=0} = \infty$ and so $\mathcal{L}\{\frac{1}{t^2}\}$ cannot be found as suggested. (Moreover, $\mathcal{L}\{\frac{1}{t^2}\}$ doesn't exist. Why?)

1) $\sin\frac{\pi}{6} = \cos\frac{\pi}{3} = .500000$

$\frac{\pi}{6} = .523598$

$\sin X = X - \frac{1}{3!}X^3 + \frac{1}{5!}X^5 \ldots \frac{1}{n!}X^n$

APPROXIMATION (TO 5 DEC)

1) $\sin\frac{\pi}{6} \cong .523598$

$\frac{1}{3!}\left(\frac{\pi}{6}\right)^3 = .023925$

2) $\sin\frac{\pi}{6} \cong .499673$

$\frac{1}{5!}\left(\frac{\pi}{6}\right)^5 = .000328$

3) $\sin\frac{\pi}{6} \cong .500002 = \cos\frac{\pi}{3}$

2) $\sin\frac{\pi}{3} = \cos\frac{\pi}{6} = .8660254$

$\frac{\pi}{3} = 1.04720$

APPROXIMATION (TO 5 DEC)

1) $\sin\frac{\pi}{3} \cong 1.04720$

$\frac{1}{3!}\left(\frac{\pi}{3}\right)^3 = .19140$

2) $\sin\frac{\pi}{3} \cong .85580$

$\frac{1}{5!}\left(\frac{\pi}{3}\right)^5 = .01049$

3) $\sin\frac{\pi}{3} \cong .86629$

$\frac{1}{7!}\left(\frac{\pi}{3}\right)^7 = .00027$

4) $\sin\frac{\pi}{3} = .86602 = \cos\frac{\pi}{6}$

2) COMPUTATION OF e $(= 2.7182818284)$

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots + \frac{1}{n!}$$

$e \cong 2.000000000$
$\frac{1}{2!} = .500000000$
$e \cong 2.500000000$
$\frac{1}{3!} = .166666667$
$e \cong 2.666666667$
$\frac{1}{4!} = .041666667$
$e \cong 2.708333333$
$\frac{1}{5!} = .008333333$
$e \cong 2.716666666$
$\frac{1}{6!} = .001388889$
$e \cong 2.718055555$
$\frac{1}{7!} = .000198413$
$e \cong 2.718253968$
$\frac{1}{8!} = .000024802$
$e \cong 2.718278770$
$\frac{1}{9!} = .000002755$
$e \cong 2.718281525$
$\frac{1}{10!} = .000000276$
$e \cong 2.718281801$
$\frac{1}{11!} = .000000025$
$e \cong 2.718281826$
$\frac{1}{12!} = .000000002$
$e \cong 2.718281828$
$\frac{1}{13!} = .000000000$
$e = 2.718281828$

Pg 654-5

1) $Y = \lim_{t \to 0} \dfrac{1 - \cos t - \frac{1}{2}t^2}{t^4} \to \dfrac{0}{0}$

$Y = \lim_{t \to 0} \dfrac{\sin t - t}{4t^3} \to \dfrac{0}{0}$

$Y = \lim_{t \to 0} \dfrac{\cos t - 1}{12t^2} \to \dfrac{0}{0}$

$Y = \lim_{t \to 0} \dfrac{-\sin t}{24t} \to \dfrac{0}{0}$

$Y = \lim_{t \to 0} \dfrac{-\cos t}{24} = \dfrac{-1}{24}$

2) $Y = \lim_{t \to 0} \dfrac{(4+h)^{\frac{1}{2}} - 2}{h} \to \dfrac{0}{0}$

$= \lim_{t \to 0} \dfrac{\frac{1}{2}(4+h)^{-1/2}}{1} = \dfrac{1}{4}$

4) $Y = \lim_{x \to 0} \dfrac{\sinh x - \sin x}{x^3} \to \dfrac{0}{0}$

$= \lim_{x \to 0} \dfrac{\cosh x - \cos x}{3x^2} \to \dfrac{0}{0}$

$= \lim_{x \to 0} \dfrac{\sinh x + \sin x}{6x} \to \dfrac{0}{0}$

$= \lim_{x \to 0} \dfrac{\cosh x + \cos x}{6} = \dfrac{1}{3}$

15) $Y = \lim_{x \to 1^+} \left(\dfrac{1}{x-1} - \dfrac{1}{\sqrt{x-1}} \right) \to \infty - \infty$

$= \lim_{x \to 1^+} \dfrac{(x-1)^{\frac{1}{2}} - (x-1)}{(x-1)^{3/2}} \to \dfrac{0}{0}$

$= \lim_{x \to 1^+} \dfrac{\frac{1}{2}(x-1)^{-1/2} - 1}{\frac{3}{2}(x-1)^{\frac{1}{2}}} \to \dfrac{\infty}{0}$

$= \lim_{x \to 1^+} \left(\dfrac{\frac{1}{2}(x-1)^{-\frac{1}{2}}}{\frac{3}{2}(x-1)^{\frac{1}{2}}} - \dfrac{1}{\frac{3}{2}(x-1)^{\frac{1}{2}}} \right) = \lim_{x \to 1^+} \frac{1}{3}(x-1)^{-\frac{1}{4}} - \frac{2}{3}(x-1)^{-\frac{1}{2}} \to \infty - \infty$

$= \lim_{x \to 1^+} \left[\left(\frac{1}{2}(x-1)^{-\frac{1}{2}} - 1 \right) \left\{ \frac{3}{2}(x-1)^{-\frac{1}{2}} \right\} \right] = \infty \cdot \infty = +\infty$

~~X~~) FIND R AND S SO THAT:

$w = \lim_{x \to 0} (x^{-3} \sin 3x + RX^{-2} + S) = 0$

LET $Y = \lim_{x \to 0} (x^{-3} \sin 3X + RX^{-2}) = {}^{-}S$

$Y = \lim_{x \to 0} (x^{-3} \sin 3X + RX^{-2}) \to \infty - \infty$

LET $Z = \frac{\sin 3X}{X^3} \to \frac{0}{0}$ AS $X \to 0$.

$\lim_{x \to 0} Z = \frac{3 \cos 3X}{3X^2} = \frac{\cos 3X}{X^2} \to \frac{1}{\infty} = 0$

$\therefore 0 = \lim_{x \to 0} \left(\frac{R}{X^2} + S \right) \Rightarrow X = \sqrt{\frac{-R}{S}}$

NUT Z!

Pg. 642

4) $R_n(x,a) = \int_a^x \frac{(x-t)^n}{n!} f^{n+1}(t)\,dt$

$= f^{n+1}(c)\,\frac{(x-a)^{n+1}}{(n+1)!}$

$n=4$; $a=0$; $|f^{n+1}(c)|_{max} = 1$

$5 \times 10^{-4} \leq \frac{x^5}{120}$

$x \geq \sqrt[5]{.06}$

2) $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \ldots \frac{f^n(a)(x-a)^n}{n!} + R_n(x,a)$

$a=0$

$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + R_n(x,0)$

$R_n(x,0) = \int_0^x \frac{(x-t)^n}{n!} f^{n+1}(t)\,dt$

$= \frac{1}{n!}\int_0^x (x-t)^n e^t\,dt$

$e^t < e^x < 3^x$

$R_n(x,0) \leq \frac{1}{n!}\int_0^x (x-t)^n 3^x\,dt$

$\leq \frac{3^x}{n!}\left(-\frac{(x-t)^{n+1}}{(n+1)}\right)\Big|_0^x$

$\leq \frac{3^x}{n!}\frac{x^{n+1}}{n+1} = \frac{3^x x^{n+1}}{(n+1)!}$ let $x=1$

$R_n(1,0) \leq \frac{3}{(n+1)!}$ $\lim_{n\to\infty} \frac{3}{(n+1)!} = 0$ CONV.

let $n=8$

$\frac{3}{9!} = .83 \times 10^{-5}$

USE $n=8$

1. Harmonic [illegible] $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges [illegible]

2. Geometric Series: $\sum_{n=0}^{\infty} a r^{n}$ converges to $\frac{a}{1-r}$ for $|r| < 1$
diverges otherwise.

3. p-Series: $\sum_{n=1}^{\infty} \frac{1}{n^{p}}$ converges for $p > 1$ and diverges for $p \le 1$.

C. **Tests for Convergence**

1. Cauchy Ratio Test (for positive-termed series)
2. Cauchy Integral Test (for positive-termed series)
3. Comparison Test (for positive-termed series)
4. Leibniz Alternating Series Test

D. **Absolute Convergence, Conditional Convergence**

1. Definition of [illegible]
2. Thm: Absolute convergence implies convergence (but not conversely)
3. Tests for absolute convergence: same as 1, 2, and 3 listed above.

E. **Power Series:** $\sum_{n=0}^{\infty} a_n (x-x_0)^n$

1. Behavior of: Determination of the interval of convergence and radius of convergence. (Cauchy's Ratio Test is [illegible] tool for this.)
2. Properties of power series:

a. If $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ and $\sum_{n=0}^{\infty} b_n (x-x_0)^n$ both converge for $|x-x_0| < R$,

then $\sum_{n=0}^{\infty} a_n (x-x_0)^n \pm \sum_{n=0}^{\infty} b_n (x-x_0)^n = \sum_{n=0}^{\infty} (a_n \pm b_n)(x-x_0)^n$.

b. i. If $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ converges at x_1, then the series converges absolutely for $|x-x_0| < |x_1-x_0|$.

ii. If $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ diverges at x_2, then the series diverges for $|x-x_0| > |x_2-x_0|$.

c. If $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ converges for $|x-x_0| < R$, then

[illegible]

d. For any numbers [illegible] in the interval $|x-x_0| < R$:

[illegible]

Observe: Special [illegible]

[illegible]

[illegible] converges for $|x-x_0| < R$, then for any positive integer [illegible] $(x-x_0)^N$ (NOTE that [illegible] expressed in powers of $(x-x_0)$ where x_0 is the same as that in the power series) then

$$[illegible] = \sum_{0}^{\infty} c_N a_n (x-x_0)^{n+N}$$

$$= \sum_{0}^{\infty} c_N a_n (x-x_0)^{n+N} \quad \text{for } |x-x_0| < R$$

[illegible] If $\sum_{0}^{\infty} a_n (x-x_0)^n$ and $\sum_{0}^{\infty} b_n (x-x_0)^n$ both converge for $|x-x_0| < R$ then

$$\left[\sum_{0}^{\infty} a_n (x-x_0)^n\right]\left[\sum_{0}^{\infty} b_n (x-x_0)^n\right] = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n} a_k b_{n-k}\right](x-x_0)^n$$

Taylor's Theorem

1. Theorem: If [illegible] $f(x)$, ..., $f^{(n)}(x)$, and $f^{(n+1)}(x)$ are all [illegible] for $|x-x_0| < R$, then

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + R_n(x, x_0)$$

where $R_n(x, x_0) = $ [illegible] (Taylor [illegible])

$$= \frac{f^{(n+1)}(\xi)(x-x_0)^{n+1}}{(n+1)!} \text{ where } x_0 < \xi < x \text{ [illegible]}$$

Taylor Series

[illegible] has derivatives of all orders at x_0, then the Taylor series generated by $f(x)$ is (by definition) $\sum_{0}^{\infty} \frac{f^{(k)}(x_0)(x-x_0)^k}{k!}$

1. If the Taylor Series generated by $f(x)$ converges for $|x-x_0| < R$ and if [illegible] $R_n(x, x_0) \to 0$ for all x in the interval of [illegible], then the Taylor Series converges [illegible] we write

$$f(x) = \sum_{0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \quad \text{for } |x-x_0| < R$$

[illegible]

b. $\sin u = \sum_{n=0}^{\infty} (-1)^n \dfrac{u^{2n+1}}{(2n+1)!}$ converges for all values of u.

c. $\cos u = \sum_{n=0}^{\infty} (-1)^n \dfrac{u^{2n}}{(2n)!}$ converges for all values of u.

d. $\dfrac{1}{1-u} = \sum_{n=0}^{\infty} u^n$ converges for $|u| < 1$.

$\dfrac{1}{1+u} = \sum_{n=0}^{\infty} (-1)^n u^n$ converges for $|u| < 1$.

e. $\sinh u = \sum_{n=0}^{\infty} \dfrac{u^{2n+1}}{(2n+1)!}$ converges for all values of u.

f. $\cosh u = \sum_{n=0}^{\infty} \dfrac{u^{2n}}{(2n)!}$ converges for all values of u.

Robert J. Marks

Pg 153-4

2) $\mathcal{L}\{t^2-3t+5\} = \int_0^\infty e^{-st}(t^2-3t+5)\,dt$

$u = t^2-3t+5 \qquad dv = e^{-st}dt$

$du = (2t-3)dt \qquad v = \frac{-1}{s}e^{-st}$

$\mathcal{L}\{t^2-3t+5\} = \frac{-1}{s}e^{-st}(t^2-3t+5) + \int_0^\infty \frac{1}{s}(2t-3)e^{-st}dt$

$u = 2t-3 \qquad dv = \frac{1}{s}e^{-st}$

$du = 2dt \qquad v = \frac{-1}{s^2}e^{-st}$

$\mathcal{L}\{t^2-3t+5\} = \frac{-1}{s}e^{-st}(t^2-3t+5) - \frac{1}{s^2}e^{-st}(2t+3) + \frac{2}{s^2}\int_0^\infty e^{-st}dt$

$= \left[\frac{-1}{s}e^{-st}(t^2-3t+5) - \frac{1}{s^2}e^{-st}(2t+3) - \frac{2}{s^3}e^{-st}\right]_0^\infty$

$= -e^{-st}\left[\frac{1}{s}(t^2-3t+5) + \frac{1}{s^2}(2t+3) + \frac{2}{s^3}\right]\Big|_0^\infty$

$= \frac{-e^{-st}}{s}(t^2-3t+5)\Big|_0^\infty - \frac{e^{-st}}{s^2}(2t+3)\Big|_0^\infty - \frac{2e^{-st}}{s^3}\Big|_0^\infty$

$= \frac{1(5)}{s} - \frac{3}{s^2} + \frac{2}{s^3}$

$= \frac{5}{s} - \frac{3}{s^2} + \frac{2}{s^3}$

3) $\mathcal{L}\{\frac{1}{2}t^3+t^2-1\} = \int_0^\infty e^{-st}(\frac{1}{2}t^3+t^2-1)dt$

$u = \frac{1}{2}t^3+t^2-1 \qquad dv = e^{-st}dt$

$du = (\frac{3}{2}t^2+2t)dt \qquad v = \frac{-1}{s}e^{-st}$

$\mathcal{L}\{\frac{1}{2}t^3+t^2-1\} = \frac{-1}{s}e^{-st}(\frac{1}{2}t^3+t^2-1)\Big|_0^\infty + \int_0^\infty \frac{1}{s}e^{-st}(\frac{3}{2}t^2+2t)dt$

$u = \frac{3}{2}t^2+2t \qquad dv = \frac{1}{s}e^{-st}dt$

$du = (3t+2)dt \qquad v = \frac{-1}{s^2}e^{-st}$

$\mathcal{L}\{\frac{1}{2}t^3+t^2-1\} = \frac{-1}{s}e^{-st}(\frac{1}{2}t^3+t^2-1)\Big|_0^\infty - \frac{1}{s^2}e^{-st}(\frac{3}{2}t^2+2t)\Big|_0^\infty + \int_0^\infty \frac{1}{s^2}e^{-st}(3t+2)dt$

$u = 3t+2 \qquad dv = (\frac{1}{s^2}e^{-st})dt$

$du = 3dt \qquad v = \frac{-1}{s^3}e^{-st}$

$\mathcal{L}\{\frac{1}{2}t^3+t^2-1\} = \frac{-1}{s}e^{-st}(\frac{1}{2}t^3+t^2-1)\Big|_0^\infty - \frac{1}{s^2}e^{-st}(\frac{3}{2}t^2+2t)\Big|_0^\infty - \frac{1}{s^3}e^{-st}(3t+2)\Big|_0^\infty - \frac{3}{s^4}e^{-st}\Big|_0^\infty$

$= \frac{-1}{s} + \frac{2}{s^3} + \frac{3}{s^4}$

4) $\mathcal{L}\{e^{-4t} + 3e^{-2t}\} = \mathcal{L}\{e^{-4t}\} + \mathcal{L}\{3e^{-2t}\}$

$$\mathcal{L}\{e^{-4t}\} = \int_0^\infty (e^{-st}\,e^{-4t})\,dt$$
$$= \int_0^\infty e^{-(s+4)t}\,dt$$
$$= \left[\frac{e^{-(s+4)t}}{-(s+4)}\right]_0^\infty$$
$$= \frac{-1}{s+4}$$

$$\mathcal{L}\{3e^{-2t}\} = 3\int_0^\infty (e^{-st}\,e^{-2t})\,dt$$
$$= 3\int_0^\infty e^{-(s+2)t}$$
$$= \left[\frac{3e^{-(s+2)t}}{-(s+2)}\right]_0^\infty$$
$$= \frac{-3}{s+2}$$

$$\therefore \mathcal{L}\{e^{-4t} + 3e^{-2t}\} = \frac{-3}{(s+2)} - \frac{1}{(s+4)}$$
$$= \frac{-3(s+4) - (s+2)}{(s+2)(s+4)} = \frac{-3s - 12 - s - 2}{(s+2)(s+4)}$$
$$= \frac{+(4s+14)}{(s+2)(s+4)}$$
$$= 2(2s+7)/(s+2)(s+4)$$

5) $\mathcal{L}\{\cosh kt\} = \mathcal{L}\{\tfrac{1}{2}e^{-t} + \tfrac{1}{2}e^{t}\}$

$$= \mathcal{L}\{\tfrac{1}{2}e^{-t}\} + \mathcal{L}\{\tfrac{1}{2}e^{t}\}$$

$$\mathcal{L}\{\tfrac{1}{2}e^{-t}\} = \frac{1}{2}\int_0^\infty e^{st}e^{-t}\,dt$$
$$= \frac{1}{2}\int_0^\infty e^{(s-1)t}\,dt$$
$$= \left[\frac{e^{(s-1)t}}{2(s-1)}\right]_0^\infty$$
$$=$$

6) $\mathcal{L}\{\cosh kt\} = \mathcal{L}\{\frac{1}{2}e^{-kt}\} + \mathcal{L}\{\frac{1}{2}e^{kt}\}$

$$\mathcal{L}\{\tfrac{1}{2}e^{-kt}\} = \frac{1}{2}\int_0^\infty (e^{-st}e^{-kt})dt$$
$$= \frac{1}{2}\int_0^\infty e^{-(s+k)t}dt$$
$$= \left[\frac{e^{-(s+k)t}}{-2(s+k)}\right]_0^\infty$$
$$= \frac{1}{2(s+k)}$$

$$\mathcal{L}\{\tfrac{1}{2}e^{kt}\} = \frac{1}{2}\int_0^\infty (e^{-st}e^{kt})dt$$
$$= \frac{1}{2}\int_0^\infty e^{(k-s)t}dt$$
$$= \left[\frac{e^{(k-s)t}}{2(k-s)}\right]_0^\infty$$
$$= \frac{1}{2(k-s)}$$

$$\mathcal{L}\{\cosh kt\} = \frac{1}{2(k-s)} + \frac{1}{2(k+s)} = \frac{s}{s^2+k^2}$$

7) $\mathcal{L}\{\cosh kt\} = \mathcal{L}\{\frac{1}{2}e^{-kt}\} - \mathcal{L}\{\frac{1}{2}e^{kt}\}$

FROM ABOVE:

$$\mathcal{L}\{\tfrac{1}{2}e^{-kt}\} = \frac{1}{2(s+k)}$$
$$\mathcal{L}\{\tfrac{1}{2}e^{kt}\} = \frac{1}{2(k-s)}$$

$$\therefore \mathcal{L}\{\cosh kt\} = \frac{1}{2(s+k)} - \frac{1}{2(k-s)}$$
$$= \frac{k-s-s-k}{2(s+k)(k-s)}$$
$$= \frac{-s}{k^2-s^2}$$
$$= \frac{s}{s^2-k^2}$$

~~8)~~ $\cos^2 A = \frac{1}{2}(1+\cos 2A)$

$\mathcal{L}\{\cos^2 A\} = \mathcal{L}\{\frac{1}{2}\} + \mathcal{L}\{\frac{1}{2}\cos 2A\}$

$\mathcal{L}\{\frac{1}{2}\} = \int_0^\infty \frac{1}{2} e^{-st}\, dt$

$= \left[\frac{-1 e^{-st}}{s2}\right]_0^\infty$

$= \frac{-1}{2s}$

$\mathcal{L}\{\frac{1}{2}\cos 2t\} = \frac{1}{2}\int_0^\infty e^{-st} \cos 2k\, dt$

$z = \int e^{-st} \cos 2k\, dt$

$U = \cos 2k \quad dV = e^{-st}\, dt$

$dU = -2\sin 2k \quad V = \frac{e}{-1}$

~~8)~~ $\cos^2 t = \frac{1}{2}\{1 + \cos 2A\}$

$\mathcal{L}\{\cos^2 t\} = \mathcal{L}\{\frac{1}{2}\} + \mathcal{L}\{\frac{1}{2}\cos 2t\}$

$\mathcal{L}\{\cos^2 kt\} = \mathcal{L}\{\frac{1}{2}\} + \mathcal{L}\{\frac{1}{2}\cos 2kt\}$

$$\mathcal{L}\{\tfrac{1}{2}\} = \int_0^\infty e^{-st}\tfrac{1}{2}\,dt$$

$$= \left[\frac{e^{-st}}{-2s}\right]_0^\infty$$

$$= \frac{-1}{+2s}$$

$\mathcal{L}\{\frac{1}{2}\cos(2k)t\} = \frac{s}{2(s^2+4k^2)}$ → FROM EQ. 5

$$\mathcal{L}\{\cos^2 kt\} = \frac{s}{2(s^2+4k^2)} - \frac{1}{2s}$$

$$= \frac{\cancel{2}(s^2 - s^2 - 4k^2)}{\cancel{2}(s^2+4k^2)(2s)}$$

$$= \frac{-4k^2}{(s^2+4k^2)(2s)}$$

11) $\sin kt \cos kt = \frac{1}{2}\sin 2kt$

$\mathcal{L}\{\frac{1}{2}\sin kt\} = \int_0^\infty \frac{1}{2}e^{-st}\sin 2kt\,dt$

$$= \frac{1}{2}\left(\frac{2k}{s^2+4k^2}\right) = \frac{k}{s^2+4k^2}$$

12) $\mathcal{L}\{e^{-at} - e^{-bt}\} = \mathcal{L}\{e^{-at}\} - \mathcal{L}\{e^{-bt}\}$

$$\mathcal{L}\{e^{-at}\} = \int_0^\infty e^{-st}e^{-at}\,dt$$

$$= \int_0^\infty e^{-(s+a)t}\,dt$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty$$

$$= \frac{1}{-(s+a)} \Rightarrow \mathcal{L}\{e^{-bt}\} = \frac{1}{-(s+b)}$$

$$\mathcal{L}\{e^{-at} - e^{-bt}\} = \frac{1}{-(s+a)} + \frac{1}{s+b}$$

$$= \frac{s+b-s-a}{(s+b)(s+a)} = \frac{b-a}{(s+b)(s+a)}$$

13) $\mathcal{L}\{\psi(t)\}$ $\quad$ $\psi(t) = 4 \quad 0<t<1$
$= 3 \quad t>1$

$$\mathcal{L}\{\psi(t)\} = 4\int_0^1 e^{-st}\,dt + 3\int_1^\infty e^{-st}\,dt$$
$$= \left[\frac{4e^{-st}}{-s}\right]_0^1 - \left[\frac{3e^{-st}}{s}\right]_0^\infty$$
$$= \frac{-4e^{-s}}{+s} + \frac{4}{s} + \frac{3e^{-st}}{s}$$
$$= \frac{1}{s}\left(4 - e^{-s}\right), \quad s>0$$

15) $\mathcal{L}\{A(t)\}$ $\quad$ $A(t) = 0 \quad 0<t<1$
$= t \quad 1<t<2$
$= 0 \quad t>0$

$$\mathcal{L}\{A(t)\} = \int_0^1 (0)\,dt + \int_1^2 e^{-st}\,t\,dt + \int_2^\infty (0)\,dt$$
$$\mathcal{L}\{A(t)\} = \mathcal{L}\{t\}_1^2 = \int_1^2 te^{-st}\,dt$$
$$dU = e^{-st} \qquad V = t$$
$$U = \frac{-e^{-st}}{s} \qquad dV = dt$$
$$\mathcal{L}\{t\}_1^2 = \left[\frac{-te^{-st}}{s}\right]_1^2 + \int_1^2 (e^{-st}/s)\,dt$$
$$= \left[\frac{-te^{-st}}{s}\right]_1^2 - \left[\frac{e^{-st}}{s^2}\right]_1^2$$
$$= \left[\frac{-2e^{-2s}}{s} + \frac{e^{-s}}{s}\right] + \left[\frac{-e^{-2s}}{s^2} + \frac{e^{-s}}{s^2}\right]$$
$$= -e^{-2s}\left[\frac{2}{s} + \frac{1}{s^2}\right] + e^{-s}\left[\frac{1}{s} + \frac{1}{s^2}\right]$$

16) $\mathcal{L}\{B(t)\}$ $\quad B(t) = \sin 2t \quad 0<t<\pi$

$\quad\quad = 0 \quad t > \pi$

$$\mathcal{L}\{B(t)\} = \int_0^{\pi} e^{-st}\sin 2t\,dt + \int_{\pi}^{\infty}(0)\,dt$$

$$= \left[\frac{e^{-st}(-s\sin 2t - 2\cos 2t)}{s^2+4}\right]_0^{\pi}$$

$$= \frac{e^{-\pi s}(+2)}{s^2+4}$$

Pg 157

1) $Y' = e^t$; $Y(0) = 1$

$\mathcal{L}\{Y'\} = \mathcal{L}\{e^t\}$

$s\,\mathcal{L}\{Y(t)\} - Y(0) = \frac{1}{s-1}$

$\mathcal{L}\{Y(t)\} = \left(\frac{1}{s-1} + 1\right)\frac{1}{s}$

$= \frac{1}{s(s-1)} + \frac{1}{s}$

$= \frac{1+s-1}{s(s-1)}$

$= \frac{1}{s-1}$

$Y = e^t$

3) $Y' - Y = 3e^t$; $Y(0) = 1$

$\mathcal{L}\{Y'\} - \mathcal{L}\{Y\} = 3\,\mathcal{L}\{e^t\}$

$s\,\mathcal{L}\{Y\} - Y(0) - \mathcal{L}\{Y\} = \frac{3}{s-1}$

$\mathcal{L}\{Y\}(s-1) = \frac{3}{(s-1)} + 1$

$= \frac{3+s-1}{(s-1)}$

$\mathcal{L}\{Y\} = \frac{s+2}{(s-1)^2}$

$Y = \sinh t$

5) $Y'' + k^2 Y = 0$; $Y(0) = 1$; $Y'(0) = 0$

$\mathcal{L}\{Y''\} + k^2\,\mathcal{L}\{Y\} = 0$

$s^2\mathcal{L}\{Y\} - sY(0) - Y'(0) + k^2\mathcal{L}\{Y\} = 0$

$s^2\mathcal{L}\{Y\} - s + k^2\mathcal{L}\{Y\} = 0$

$\mathcal{L}\{Y\}(s^2 + k^2) = s$

$\mathcal{L}\{Y\} = \frac{s}{s^2 + k^2}$

$Y = \cos kt$

7) $Y'' - 3Y' + 2Y = e^{3t}$; $Y(0) = Y'(0) = 0$

$\mathcal{L}\{Y''\} - 3\mathcal{L}\{Y'\} + 2\,\mathcal{L}\{Y\} = \mathcal{L}\{e^{3t}\}$

$s^2\mathcal{L}\{Y\} - sY(0) - Y''(0) - 3s\,\mathcal{L}\{Y\} + 3Y(0) + 2\mathcal{L}\{Y\} = \frac{1}{s-3}$

$s^2\mathcal{L}\{Y\} - 3s\,\mathcal{L}\{Y\} + 2\,\mathcal{L}\{Y\} = \frac{1}{s-3}$

$\mathcal{L}\{Y\}(s^2 - 3s + 2) = \frac{1}{s-3}$

$\mathcal{L}\{Y\} = \frac{1}{(s-3)(s-2)(s-1)}$

$\frac{1}{(s-3)(s-2)(s-1)} = \frac{A}{s-3} + \frac{B}{s-2} + \frac{C}{s-1}$

$A = \frac{1}{2}$; $B = -1$; $C = \frac{1}{2}$

$\therefore\ Y = \frac{1}{2}e^{3t} - e^{2t} + \frac{1}{2}e^t$

~~9)~~ $Y'' - 2Y' = -4$; $Y(0) = 0$; $Y'(0) = 4$

$s^2\mathcal{L}\{Y\} - sY(0) - Y'(0) - 2\mathcal{L}\{Y\} + 2Y(0) = \mathcal{L}\{-4\}$

$\mathcal{L}\{Y\}(s^2 - 2\} - 4 \qquad = \frac{-4}{s}$

$\mathcal{L}\{Y\} = \left(\frac{-4}{s} + 4\right)\frac{1}{s^2 - 2}$

$= \left(\frac{-4 + 4s}{s}\right)\left(\frac{1}{s^2 - 2}\right)$

$= \frac{4s - 4}{s(s^2 - 2)} = \frac{4(s-1)}{s(s^2 - 2)}$

9) $Y'' - 2Y' = -4$; $Y(0) = 2$; $Y'(0) = 3$

$s^2\mathcal{L}\{Y\} - sY(0) - Y'(0) - 2\mathcal{L}\{Y\} + 2Y(0) = \frac{-4}{s}$

$s^2\mathcal{L}\{Y\} - 2s - 3 - 2\mathcal{L}\{Y\} + 4 = \frac{-4}{s}$

$\mathcal{L}\{Y\}(s^2 - 2) = \frac{-4}{s} + 2s + 1$

$= \frac{-4 + 2s^2 + s}{s} = \frac{2s^2 + s - 4}{s}$

$\mathcal{L}\{Y\} = \frac{2}{s} + \frac{1}{s-1} - \frac{1}{s+2}$

$Y = 2 + e^{t} - e^{-2t}$

2) $\mathcal{L}^{-1}\left\{\frac{1}{s^2-6s+10}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2+1}\right\}$

$= e^{3t}\mathcal{L}^{-1}\frac{1}{s^2+1}$

$= e^{3t}\sin t$

3) $\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{(s+1)^2+3}\right\}$

$= e^{-t}\mathcal{L}^{-1}\left\{\frac{s-1}{s^2+3}\right\}$

$= e^{-t}\left[\mathcal{L}^{-1}\left\{\frac{s}{s^2+3}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+3}\right\}\right]$

$= e^{-t}\left(\cos 2t - \frac{1}{2}\sin 2t\right)$

4) $\mathcal{L}^{-1}\left\{\frac{s}{s^2-6s+13}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{(s-3)^2+4}\right\}$

$= e^{3t}\mathcal{L}^{-1}\frac{s-3}{s^2+4}$

$= e^{3t}\left[\mathcal{L}^{-1}\frac{s}{s^2+4} - \mathcal{L}^{-1}\frac{3}{s^2+4}\right]$

$= e^{3t}\left[\mathcal{L}^{-1}\frac{s}{s^2+4} - \frac{4}{3}\mathcal{L}^{-1}\frac{4}{s^2+4}\right]$

$= e^{3t}\left(\cos 2t - \frac{4}{3}\sin 2t\right)$

7) $\mathcal{L}^{-1}\frac{(s-5)}{s^2+6s+13} = \mathcal{L}^{-1}\frac{s-5}{(s+3)^2+4}$

$= e^{-3t}\mathcal{L}^{-1}\left\{\frac{s-2}{s^2+4}\right\}$

$= e^{-3t}\left[\mathcal{L}^{-1}\frac{s}{s^2+4} - \mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}\right]$

$= e^{-3t}\left[\cos 2t - 4\mathcal{L}^{-1}\frac{4}{s^2+4}\right]$

$= e^{-3t}\left(\cos 2t - 4\sin 2t\right)$

8) $\mathcal{L}^{-1}\frac{2s-1}{s^2+4s+29} = \mathcal{L}^{-1}\left\{\frac{2s-1}{(s+2)^2+25}\right\}$

$= e^{-2t}\mathcal{L}^{-1}\left\{\frac{2(s+2)-1}{s^2+25}\right\}$

$= e^{-2t}\mathcal{L}^{-1}\left\{\frac{2s-3}{s^2+25}\right\}$

$= e^{-2t}\left[\mathcal{L}^{-1}\frac{2s}{s^2+25} - \mathcal{L}^{-1}\frac{3}{s^2+25}\right]$

$= e^{-2t}\left[2\mathcal{L}^{-1}\frac{s}{s^2+25} - \frac{3}{25}\mathcal{L}^{-1}\frac{25}{s^2+25}\right]$

$= e^{-2t}\left(2\cos 5t - \frac{3}{25}\sin 5t\right)$

FROM PHOTOSTAT

1) $\mathcal{L}^{-1}\left\{\frac{1}{s^2+as}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s(s+a)}\right\}$

$\frac{1}{s(s+a)} \equiv \frac{A}{s} + \frac{B}{s+a}$

$1 \equiv A(s+a) + Bs$

$s=0 \Rightarrow 1 = Aa \Rightarrow A = \frac{1}{a}$

$s=-a \Rightarrow 1 = -Ba \Rightarrow B = \frac{-1}{a}$

$\therefore \mathcal{L}^{-1}\left\{\frac{1}{s^2+as}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{as} - \frac{1}{a(s+a)}\right\}$

$= \frac{1}{a} - \frac{1}{a}e^{-at}$

$= \frac{1}{a}\left(1 - e^{-at}\right)$

3) $\mathcal{L}^{-1}\left\{\frac{2s^2+5s-4}{s^3+s^2-2s}\right\} = \mathcal{L}^{-1}\left\{\frac{2s^2+5s-4}{s(s+2)(s-1)}\right\}$

$\frac{2s^2+5s-4}{s(s+2)(s-1)} \equiv \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s-1}$

$2s^2+5s-4 \equiv A(s+2)(s-1) + B(s-1)s + Cs(s+2)$

$s=0 \Rightarrow -4 = -2A \Rightarrow A = 2$

$s=1 \Rightarrow 3 = 3C \Rightarrow C = 1$

$s=-2 \Rightarrow -6 = -3B \Rightarrow B = 2$

$\therefore \mathcal{L}^{-1}\left\{\frac{2s^2+5s-4}{s^3+s^2-2s}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{s} + \frac{2}{s+2} + \frac{1}{s-1}\right\}$

$= 2 + 2e^{-2t} + e^{t}$

5) $\mathcal{L}^{-1}\left\{\frac{4s+4}{s^2(s-2)}\right\}$

$\frac{4s+4}{s^2(s-2)} \equiv \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-2}$

$4s+4 \equiv As(s-2) + B(s-2) + Cs^2$

$s=2 \Rightarrow 12 = 4C \Rightarrow C = 3$

$s=0 \Rightarrow 4 = -2B \Rightarrow B = -2$

$4s+4 \equiv As(s-2) - 2(s-2) + 3s^2$

$4s+4 \equiv As^2 - 2As - 2s + 4 + 3s^2$

$4s+4 \equiv s^2(A+3) - s(2A+2) + 4 \Rightarrow A = -3$

$\therefore \mathcal{L}^{-1}\left\{\frac{4s+4}{s^2(s-2)}\right\} = \mathcal{L}^{-1}\left\{\frac{-3}{s} - \frac{2}{s^2} + \frac{3}{s-2}\right\}$

$= -3 - 2t + 3e^{2t}$

8) $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)(s^2+b^2)}\right\}$

$$\frac{1}{(s^2+a^2)(s^2+b^2)} = \frac{As+B}{s^2+a^2} + \frac{Cs+D}{s^2+b^2}$$

$$1 = (As+B)(s^2+b^2) + (Cs+D)(s^2+a^2)$$

$$s = ai \Rightarrow 1 = (Aai+B)(b^2-a^2)$$

$$1 = Aaib^2 - Aa^3i + Bb^2 - Ba^2$$

$$iA(ab^2-a^3) = 0 \Rightarrow A = 0$$

$$B(b^2-a^2) = 1 \Rightarrow B = \frac{1}{b^2-a^2}$$

$$s = bi \Rightarrow 1 = (Cbi+D)(a^2-b^2)$$

$$= Cbi(a^2-b^2) + D(a^2-b^2)$$

$$Cbi(a^2-b^2) = 0 \Rightarrow C = 0$$

$$D(a^2-b^2) = 1 \Rightarrow D = \frac{1}{a^2-b^2}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)(s^2+b^2)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(b^2-a^2)(s^2+a^2)} - \frac{1}{(b^2-a^2)(s^2+b^2)}\right\}$$

$$= \frac{1}{ab(b^2-a^2)}\mathcal{L}^{-1}\left\{\frac{ab}{s^2+a^2} - \frac{ab}{s^2+b^2}\right\}$$

$$= \frac{b\sin at - a\sin bt}{ab(b^2-a^2)}$$

10) $\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$

$$\frac{s^2}{(s^2+a^2)(s^2+b^2)} = \frac{As+B}{s^2+a^2} + \frac{Cs+D}{s^2+b^2}$$

$$s^2 = (As+B)(s^2+b^2) + (Cs+D)(s^2+a^2)$$

$$s = bi \Rightarrow -b^2 = (Cbi+D)(a^2-b^2)$$

$$= Cbi(a^2-b^2) + D(a^2-b^2)$$

$$Cbi(a^2-b^2) = 0 \Rightarrow C = 0$$

$$D(a^2-b^2) = -b^2 \Rightarrow D = \frac{b^2}{b^2-a^2}$$

$$s = ai \Rightarrow -a^2 = (Aai+B)(b^2-a^2)$$

$$= Aai(b^2-a^2) + B(b^2-a^2)$$

$$\Rightarrow A = 0 \;;\; B = \frac{a^2}{a^2-b^2}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\} = \mathcal{L}^{-1}\left\{\frac{a^2}{(a^2-b^2)(s^2+a^2)} - \frac{b^2}{(a^2-b^2)(s^2+b^2)}\right\}$$

$$= \frac{a\sin at - b\sin bt}{(a^2-b^2)}$$

13) $\mathcal{L}^{-1}\left\{\dfrac{s^2+2s+5}{(s^2+2s+2)(2s^2+2s+5)}\right\}$

$$\frac{s^2+2s+5}{(s^2+2s+2)(2s^2+2s+5)} = \frac{As+B}{2s^2+2s+5} + \frac{Cs+D}{2s^2+2s+2}$$

$$s^2+2s+5 \equiv (As+B)(2s^2+2s+2) + (Cs+D)(2s^2+2s+5)$$
$$\equiv 2s^2(As+B) + s2(As+B) + 2(As+B) + 2s^2(Cs+D)$$
$$+ 2s(Cs+D) + 5(Cs+D)$$
$$\equiv s^3 2A + s^2 2B + s^2 2A + s2B + s2A + 2B + s^3 2C + s^2 2D$$
$$+ s^2 2C + s2D + s5C + 5D$$
$$\equiv s^3(2A+2C) + s^2(2B+2A+2D+2C) + s(2B+2A+2D+5C)$$
$$+ (2B+5D)$$

$$\Rightarrow \quad 2A+2C=0 \quad \Rightarrow \quad A + C = 0$$
$$2B+2A+2D+2C=1 \quad \Rightarrow \quad A+B+C+D=\tfrac{1}{2}$$
$$2B+2A+2D+5C=2 \quad \Rightarrow \quad A+B+C+\tfrac{5}{2}D=1$$
$$2B+5D=5 \qquad\qquad 2B + 5D = 5$$

$$\tfrac{3}{2}D = \tfrac{1}{2} \Rightarrow D = \tfrac{1}{3}$$
$$2B + \tfrac{5}{3} = 5 \Rightarrow 2B = \tfrac{10}{3} \Rightarrow B = \tfrac{5}{3}$$
$$B + D = 2$$
$$A + C + 2 = \tfrac{1}{2} \Rightarrow 2 = \tfrac{1}{2} \quad \text{NUTZ!}$$

~~14)~~ $\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+4)^2}\right\}$

$$\frac{s^2}{(s^2+4)^2} = \frac{As+B}{s^2+4} + \frac{Cs+D}{(s^2+4)^2}$$

$$s^2 = (As+B)(s^2+4) + (Cs+D)$$

$$s \Rightarrow \sqrt{2}\,i \Rightarrow -4 = C\sqrt{2}\,i + D$$

$$C=0 \;;\; D=-4$$

$$s = \frac{-D}{C} = \infty\ ?$$

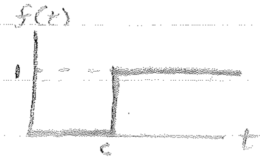
$$\frac{s^2}{(s^2+4)} = \frac{As+B}{s^2+4}$$

$$\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+4)^2}\right\} = \mathcal{L}^{-1}\left\{s\ \frac{s}{(s^2+4)^2}\right\} \quad \text{NUTZ!}$$

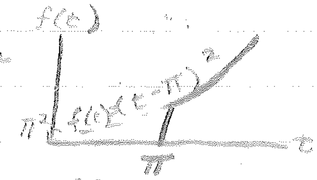
14) $\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+4)^2}\right\} = e^{-4t}\,\mathcal{L}\ \frac{(s+4)^2}{s^2}$

Pp. 182-3

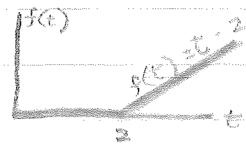
1) $\alpha(t-c)$



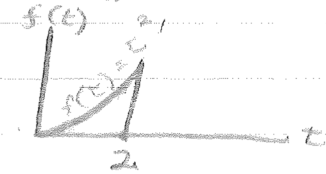
5) $(t-\pi)\alpha(t-\pi)^2$



3) $(t-2)\alpha(t-2)$

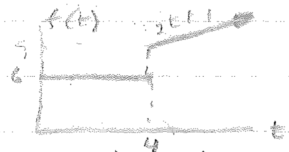


7) $t^2 - t^2\alpha(t-2)$



9) $F(t) = 6 \quad 0<t<4$

$\quad = 2t+1 \quad t>4$



$F(t) = 6\alpha t - 6\alpha(t-4) + (2t+1)\alpha(t-4)$

$\quad = \alpha(t-4)[2t-5] + 6$

$f(t-4) = 2t-5$

$f(t) = 2t+3$

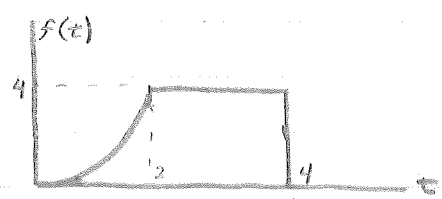
$\mathcal{L}\{f(t)\} = \frac{2}{s^2} + \frac{3}{s}$

$\mathcal{L}\{F(t)\} = \frac{6}{s} + e^{-4s}\left(\frac{2}{s^2} + \frac{3}{s}\right)$

11) $F(t) = t^2; \quad 0<t<2$

$\quad = 4 \quad 2<t<4$

$\quad = 0 \quad t>4$



$F(t) = t^2 - t^2\alpha(t-2) + 4\alpha(t-2) - 4\alpha(t-4)$

$\quad = \alpha(t-2)[4-t^2] - \alpha(t-4)4 + t^2$

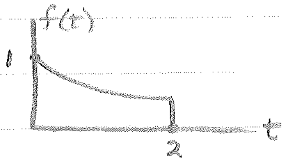
$f(t-2) = 4-t^2 \qquad g(t-4) = g(t) = 4$

$f(t) = 4-(t+2)^2 = 4 - t^2 - 4t - 4 = t^2 - 4t$

$\mathcal{L}\{F(t)\} = \frac{2}{s^3} - \frac{4}{s}e^{-4s} + e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2}\right)$

13) $F(t) = e^{-t} \quad 0 < t < 2$

$\qquad = 0 \qquad t > 2$



$F(t) = e^{-t} - e^{-t}\alpha(t-2)$

$f(t-2) = -e^{-t}$

$f(t) = +e^{-(t+2)}$

$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-(t+2)} e^{-st} dt$

$\qquad = \int_0^\infty e^{-t-2-st} dt = \int e^{-t(1+s)-2} dt$

$\mathcal{L}\{f(t)\} = \left[\frac{e^{-t(1+s)-2}}{-(s+1)} \right]_0^\infty$

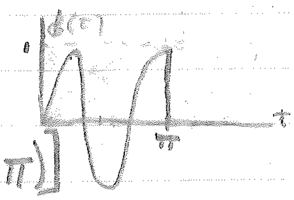
$\qquad = e^{-2}/(s+1)$

$\mathcal{L}\{F(t)\} = \frac{1}{s+1} - \frac{e^{-2}e^{-2s}}{s+1}$

$\qquad = \frac{1-e^{-2(s+1)}}{s+1}$

15) $\phi(t) = \sin 3t \quad 0 < t < \pi$

$\qquad = 0 \qquad t > \pi$



$\phi(t) = \sin 3t - \sin 3t[\alpha(t-\pi)]$

$\#(t-\pi) = \sin 3t$

$\#(t) = \sin 3(t+\pi) = \mathcal{L}\{\int_0^\infty e^{-st} \sin(3t+3\pi)\, dt\}$

$U = \sin(3t+3\pi) \qquad dV = e^{-st} dv$

$dU = 3\cos(3t+3\pi)dt \qquad V = \frac{-e^{-st}}{s}$

$\mathcal{L}\{\#(t)\} = \left[\frac{-e^{-st}\sin(3t+3\pi)}{s} \right]_0^\infty + \int_0^\infty \frac{3\cos(3t+3\pi)e^{-st}}{s} dt$

$U = 3\cos(3t+3\pi) \qquad dV = \frac{e^{-st}}{s} dv$

$dU = -9\sin(3t+3\pi)dt \qquad V = \frac{-e^{-st}}{s^2}$

$\mathcal{L}\{\#(t)\} = \left[\frac{-e^{-st}\sin(3t+3\pi)}{s} \right]_0^\infty - \left[\frac{3\cos(3t+3\pi)e^{-st}}{s^2} \right]_0^\infty$

$\qquad - \frac{9}{s^2} \int_0^\infty \sin(3t+3\pi) e^{-st} dt$

$\frac{s^2}{9}\mathcal{L}\{\#(t)\} = \frac{-s^2 \sin 3\pi}{2s} + \frac{3\cos 3\pi}{s^2}\left(\frac{s^2}{9}\right) - \int_0^\infty \sin(3t+3\pi) e^{-st} dt$

$\frac{2s^2}{9}\mathcal{L}\{\#(t)\} = \frac{1}{3}$

$\mathcal{L}\{\#(t)\} = \frac{3}{2s^2}$

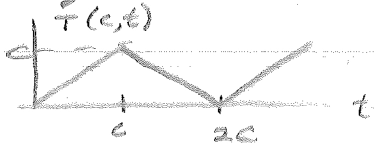
$\mathcal{L}\{\phi(t)\} = -e^{-\pi s}\left(\frac{3}{2s^2}\right) + \frac{s}{s^2+9}$

Pp. 172-3

11) $T(c,t) = t \qquad 0<t<c$

$= 2c - t \qquad c<t<2c$

$T(t,c) = T(t+2c, c)$



$T_1(t,c) = t - t\alpha(t-c) + (2c-t)\alpha(t-c) - (2c-t)\alpha(2c-t)$

$= t - 2(t-c)\alpha(t-c) + (2c+t)\alpha(t-2c)$

$= \frac{1}{s^2} - 2e^{-cs}\frac{1}{s^2} + e^{-2cs}\frac{1}{s^2}$

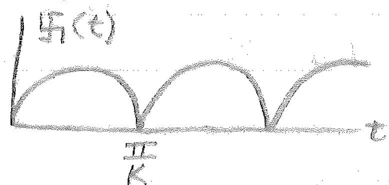
$= \frac{1}{s^2}\left[e^{-2cs} - 2e^{-cs} + 1\right]$

$= \frac{1}{s^2}(e^{-cs} - 1)^2$

$T(t,c) = \frac{1}{s^2}\frac{(e^{-cs}-1)^2}{1-e^{-2cs}} = \frac{(1-e^{-cs})^2}{s^2(1+e^{-cs})(1-e^{-sc})} = \frac{1}{s^2}\frac{(1-e^{-cs})}{(1+e^{-cs})}$

$= \frac{1}{s^2}\tanh\frac{cs}{2}$

12) $\mathfrak{H}(t) = |\sin kt|$



$\mathfrak{H}_1(t) = \sin kt - \sin kt\,\alpha(t - \frac{\pi}{k})$

$q(t - \frac{\pi}{k}) = -\sin kt$

$q(t) = -\sin k(t + \frac{\pi}{k})$

$= -\sin(kt+\pi) = \sin kt$

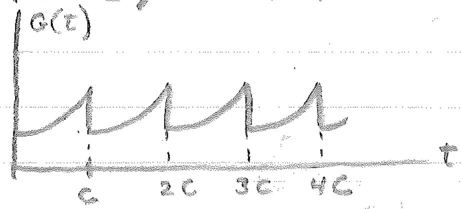
$\mathcal{L}\{q(t)\} = e^{-\frac{\pi s}{k}}\frac{k}{k^2+s^2}$

$\mathcal{L}\{\mathfrak{H}_1(t)\} = \frac{k}{k^2+s^2}(e^{-\frac{\pi s}{k}} + 1)$

$\mathcal{L}\{\mathfrak{H}(t)\} = \frac{k}{k^2+s}\left[\frac{1+e^{-\frac{\pi s}{k}}}{1-e^{-\frac{\pi s}{k}}}\right]$

15) $G(t) = e^t \qquad 0<t<c$

$G(t+c) = G(t)$



$G_1(t) = e^t - e^t \alpha(t-c)$

$g(t-c) = -e^t$

$g(t) = -e^{t+c} = -e^c e^t$

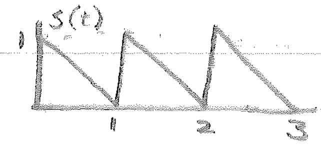
$\mathcal{L}\{g(t)\} = \frac{-e^c}{s-1}$

$\mathcal{L}\{G_1(t)\} = \frac{1}{s-1}(1-e^c)$

$\mathcal{L}\{G(t) = \frac{1}{s-1} \frac{(1-e^c)}{(1-e^{-cs})}$

16) $S(t) = 1-t \qquad 0<t<1$

$G(t+1) = G(t)$



$S_1(t) = (1-t) - (1-t)\alpha(t-1)$

$= 1 - t + (t-1)\alpha(t-1)$

$\mathcal{L}\{S_1(t)\} = \frac{1}{s} - \frac{1}{s^2} + e^{-s}\left(\frac{1}{s^2} - \frac{1}{s}\right)$

$= \left(\frac{1}{s} - \frac{1}{s^2}\right)\left(1 - e^{-s}\right)$

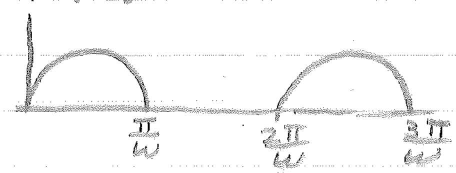
$\mathcal{L}\{S(t)\} = \left(\frac{1}{s} - \frac{1}{s^2}\right) \frac{1-e^{-s}}{1-e^{-s}}$

$= \frac{1}{s} - \frac{1}{s^2}$

17) $F(t) = \sin \omega t \qquad 0 < t < \frac{\pi}{\omega}$

$\qquad = 0 \qquad \frac{\pi}{\omega} < t < \frac{2\pi}{\omega}$

$F\left(t + \frac{2\pi}{\omega}\right) = F(t)$



$F_1(t) = \sin \omega t - \sin \omega t \, \alpha\left(t - \frac{\pi}{\omega}\right)$

$G\left(t - \frac{\pi}{\omega}\right) = -\sin \omega (t)$

$G(t) = -\sin \omega \left(t + \frac{\pi}{\omega}\right)$

$= \sin(\omega t)$

$\mathcal{L}\{G_1(t)\} = e^{-\frac{\pi s}{\omega}} \frac{\omega}{s^2 + \omega^2}$

$\mathcal{L}\{F_1(t)\} = \frac{\omega}{s^2 + \omega^2}\left(1 + e^{-\frac{\pi s}{\omega}}\right)$

$\mathcal{L}\{F(t)\} = \frac{\omega}{s^2 + \omega^2} \frac{\left(1 + e^{-\frac{\pi s}{\omega}}\right)}{\left(1 - e^{-\frac{2\pi s}{\omega}}\right)}$

$= \frac{\omega}{(s^2 + \omega)(1 + e^{-\pi s/\omega})}$

Pp. 186-7

1) $\int_0^t (t-\beta)\sin 2\beta\, d\beta = q(t)$

$u = t-\beta \quad dv = \sin 2\beta\, d\beta$

$du = -d\beta \quad v = \frac{-1}{2}\cos 2\beta$

$\left[\frac{1}{2}(\beta-t)\cos 2\beta\right]_0^t - \int_0^t \frac{1}{2}\cos 2\beta\, d\beta$

$\frac{1}{2}(-t) - \left[\frac{1}{4}\sin 2\beta\right]_0^t$

$-\frac{t}{2} - \frac{1}{4}\sin 2t$

$\mathcal{L}\{q(t)\} = \frac{-1}{2s^2} - \frac{1}{s^2+4}$

2) $\int_0^t e^{\beta-t}\cos\beta\, d\beta = \mathfrak{I}(t)$

$u = e^{\beta-t} \quad dv = \cos\beta\, d\beta$

$du = e^{\beta-t} d\beta \quad v = -\sin\beta\, d\beta$

$\mathfrak{I}(t) = \left[-e^{\beta-t}\sin\beta\right]_0^t + \int_0^t e^{\beta-t}\sin\beta\, d\beta$

$u = e^{\beta-t} \quad dv = \sin\beta\, d\beta$

$du = e^{\beta-t} \quad v = \cos\beta$

$\mathfrak{I}(t) = -\sin t + \left[\cos\beta\, e^{\beta-t}\right]_0^t - \int_0^t e^{-\beta t}\cos\beta\, d\beta$

$2\mathfrak{I}(t) = -\sin t + \cos t - e^{-t}$

$\mathcal{L}\{\mathfrak{I}(t)\} = \frac{-1}{2(s^2+1)} + \frac{s}{2(s^2+1)} - \frac{1}{s+1}$

~~3)~~ $\mathcal{C}(t) = \int_0^t (t-\beta)^2 e^\beta\, d\beta$

$= \int_0^t (t^2 - 2\beta + \beta^2) e^\beta\, d\beta$

$u = t^2 - 2\beta + \beta^2 \quad dv = e^\beta d\beta$

$du = (-2+2\beta)d\beta \quad v = e^\beta$

$\mathcal{C}(t) = \left[e^\beta(t^2 - 2\beta + \beta^2)\right]_0^t + \int_0^\beta e^\beta(2-2\beta)d\beta$

$u = 2-2\beta \quad dv = e^\beta$

$du = -2d\beta \quad v = e^\beta$

$\mathcal{C}(t) = e^t(t^2 - 2t + t^2) - t^2 + \left[e^\beta(2-2\beta\right]_0^t + \int_0^t 2e^\beta d\beta$

$= 2e^t(t^2 - t) - t^2 + e^t(2-2t) - 2 + 2e^t - 2$

3) $\mathcal{C}(t) = \int_0^t (t-\beta)^2 e^{\beta}\, d\beta$

$$U = (t-\beta)^2 \qquad dV = e^{\beta} d\beta$$

$$dU = -2(t-\beta)d\beta \qquad V = e^{\beta}$$

$$\mathcal{C}(t) = \left[e^{\beta}(t-\beta)^2\right]_0^t + \int_0^t 2e^{\beta}(t-\beta)\,d\beta$$

$$U = t-\beta \qquad dV = 2e^{\beta} d\beta$$

$$dU = -d\beta \qquad V = 2e^{\beta}$$

$$\mathcal{C}(t) = -t^2 + \left[2e^{\beta}(t-\beta)\right]_0^t + \int_0^t 2e^{\beta} d\beta$$

$$= -t^2 - 2t + \left[2e^{\beta}\right]_0^t$$

$$= -t^2 - 2t + 2e^t - 2$$

$$\mathcal{L}\{\mathcal{C}(t)\} = \frac{-2}{s^3} - \frac{2}{s^2} + \frac{2}{s-1} - \frac{2}{s}$$

$$= \frac{-2s + 2 - 2s(s-1) + 2s^3 - 2s^2(s-1)}{s^3(s-1)}$$

$$= \frac{2}{s^3(s-1)}$$

4) $f(s) = \frac{1}{s} \qquad g(s) = \frac{k}{s^2+k^2}$

$$\mathcal{L}^{-1}\{f(s)\} = 1 = F(t) \qquad \mathcal{L}^{-1}\{g(s)\} = \sin k\beta = G(\beta)$$

$$G(t-\beta) = \sin k(t-\beta)$$

$$\mathcal{L}^{-1}\{f(s)g(s)\} = \int_0^t \sin k(t-\beta)\,d\beta$$

$$= \int_0^t \sin(kt-k\beta)\,d\beta$$

$$= \left[+\frac{1}{k}\cos(kt-k\beta)\right]_0^t$$

$$= \frac{1}{k} - \frac{1}{k}\cos k(t-\beta)$$

$$= \frac{1}{k}(1-\cos k(t-\beta))$$

5) $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = F(t) = t^2 \qquad \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = G(\beta) = e^{\beta}$

$$\Psi(t) = \mathcal{L}^{-1}\{f(s)g(s)\} = \int_0^t \beta^2 e^{t-\beta} d\beta$$

$$U = \beta^2 \qquad dV = e^{t-\beta} d\beta$$

$$dU = 2\beta\, d\beta \qquad V = -e^{t-\beta}$$

$$\Psi(t) = \left[-\beta^2 e^{t-\beta}\right]_0^t + \int_0^t 2\beta\, e^{t-\beta} d\beta$$

$$U = 2\beta \qquad dV = e^{t-\beta} d\beta$$

$$dU = 2\, d\beta \qquad V = -e^{t-\beta}$$

$$\Psi(t) = -t^2 + \left[-2\beta e^{t-\beta}\right]_0^t + \int_0^t 2e^{t-\beta} d\beta$$

$$= -t^2 - 2t + \left[-2e^{t-\beta}\right]_0^t$$

$$= -t^2 - 2t - 2 + 2e^t$$

$$\mathcal{L}^{-1}\{\Psi(t)\} = \frac{-2}{s^3} - \frac{2}{s^2} - \frac{2}{s} + \frac{2}{s-1}$$

6) $b(s) = \frac{1}{(s^2+1)^2}$

$g(s) = f(s) = \frac{1}{s^2+1}$

$\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$

$G(\beta) = F(\beta) = \sin \beta$

$G(t-\beta) = \sin(t-\beta)$

$\mathcal{L}^{-1}\{b(s)\} = \int_0^t \sin(\beta)\sin(t-\beta)\,d\beta = b(t)$

$U = \sin(t-\beta) \quad dV = \sin\beta\,d\beta$

$dU = -\cos(t-\beta) \quad V = -\cos\beta$

$b(t) =$ ACH!

Pg 192

3) $Y''(t) - Y(t) = 5\sin 2t \;;\; Y(0) = 0 \;;\; Y'(0) = 1$

$s^2Y(s) - sY(0) - Y'(0) - Y(s) = \frac{20}{s^2+4}$

$Y(s)[s^2-1] = \frac{20}{s^2+4} + s$

$Y(s) = \frac{20}{(s^2+4)(s+1)(s-1)} + \frac{s}{(s+1)(s-1)}$

$Y_1(s) = \frac{s}{(s+1)(s-1)} = \frac{A}{s+1} + \frac{B}{s-1}$

$s = A(s-1) + B(s+1)$

$s = 1 \Rightarrow 1 = 2B \Rightarrow B = \frac{1}{2}$

$s = -1 \Rightarrow -1 = -2A \Rightarrow A = \frac{1}{2}$

$\therefore Y_1(s) = \frac{1}{2(s+1)} + \frac{1}{2(s-1)}$

$Y_2(s) = \frac{20}{(s^2+4)(s+1)(s-1)} = \frac{As+B}{s^2+4} + \frac{C}{s+1} + \frac{D}{s-1}$

$20 = (As+B)(s^2-1) + C(s^2+4)(s-1) + D(s^2+4)(s+1)$

$s = -1 \Rightarrow 20 = C(-10) \Rightarrow C = -\frac{1}{2}$

$s = 1 \Rightarrow 20 = D(10) \Rightarrow D = \frac{1}{2}$

$s = 2i \Rightarrow 20 = (2iA + B)(-5)$

$20 = -10iA - 5B$

$A = 0 \;;\; B = \frac{-20}{5} = -4$

$\therefore Y_2(s) = \frac{-4}{s^2+4} - \frac{1}{2(s+1)} + \frac{1}{2(s-1)}$

$Y(s) = Y_1(s) + Y_2(s)$

$= \frac{1}{2(s+1)} + \frac{1}{2(s-1)} - \frac{4}{s^2+4} - \frac{1}{2(s+1)} + \frac{1}{2(s-1)}$

$= \frac{1}{s-1} - \frac{4}{s^2+4}$

$\mathcal{L}^{-1}\{Y(s)\} = Y(t) = e^t - \sin 2t$

5) $x''(t) + 4x(t) = 2t - 8$; $x(0) = 1$; $x'(0) = 0$

$s^2\bar{X} - s + 4\bar{X} = \frac{2}{s^2} - \frac{8}{s}$

$\bar{X}(s^2+4) = \frac{2}{s^2} - \frac{8}{s} + s$

$\bar{X} = \frac{2}{s^2(s^2+4)} - \frac{8}{s(s^2+4)} + \frac{s}{s^2+4}$

$$G_1(s) = \frac{2}{s^2(s^2+4)} \equiv \frac{A}{s^2} + \frac{B}{s^2+4}$$

$$2 \equiv A(s^2+4) + Bs^2$$

$$s = 0 \Rightarrow 2 = 4A \Rightarrow A = \frac{1}{2}$$

$$s = 2i \Rightarrow 2 = {}^-4B \Rightarrow B = {}^-\frac{1}{2}$$

$$G_2(s) = \frac{-8}{s(s^2+4)} \equiv \frac{A}{s} + \frac{Bs+C}{s^2+4}$$

$$-8 \equiv A(s^2+4) + s(Bs+C)$$

$$s = 0 \Rightarrow {}^-8 = 4A \Rightarrow A = {}^-2$$

$$s = 2i \Rightarrow {}^-8 = 2i(2iB + C)$$

$$-8 = {}^-4B + 2iC$$

$$\Rightarrow B = 2 \; ; \; C = 0$$

$\bar{X} = \frac{1}{2s^2} - \frac{4}{(s^2+4)} - \frac{2}{s} + \frac{2}{s^2+4} + \frac{s}{s^2+4}$

$\mathcal{L}^{-1}\{\bar{X}\} = \frac{1}{2}t - \frac{1}{8}\sin 2t - 2 + \frac{1}{2}\sin 2t + \cos 2t$

7) $U''(t) + 4U(t) = 15e^t$; $U(0) = 0$; $U'(0) = 3$

$s^2\bar{U} - sU(0) - U'(0) + 4\bar{U} = \frac{15}{s-1}$

$s^2\bar{U} - 3 + 4\bar{U} = \frac{15}{s-1}$

$\bar{U}(s^2+4) = \frac{15}{s-1} + 3$

$\bar{U} = \frac{15}{(s-1)(s^2+4)} + \frac{3}{s^2+4}$

$$G(s) = \frac{15}{(s-1)(s^2+4)} \equiv \frac{A}{s-1} + \frac{Bs+C}{s^2+4}$$

$$15 \equiv A(s^2+4) + (s-1)(Bs+C)$$

$$s = 1 \Rightarrow 15 = A5 \Rightarrow A = 3$$

$$s = 2i \Rightarrow 15 = (2i-1)(2iB + C)$$

$$15 = {}^-4B + 2iC - 2iB - C$$

$$^-4B - C = 15 \; ; \; 2C = 2B \Rightarrow B = C$$

$$^-5B = 15 \Rightarrow B = C = {}^-3$$

$\bar{U} = \frac{3}{s-1} - \frac{3s+3+3}{s^2+4} = \frac{3}{s-1} - \frac{3(s+2)}{(s^2+4)}$

11) $x''(t) + 3x'(t) + 2x(t) = 4t^2$; $x(0) = x'(0) = 0$

$$s^2\bar{x} + 3s\bar{x} + 2\bar{x} = \frac{8}{s^3}$$

$$\bar{x} = \frac{8}{s^3(s^2+3s+2)} = \frac{8}{s^3(s+2)(s+1)}$$

$$\frac{8}{s^3(s+2)(s-3)} = \frac{A}{s^3} + \cancel{\frac{B}{s}} + \frac{C}{(s+2)} + \frac{D}{(s+1)}$$

$$8 = A(s+2)(s+1) + Cs^3(s+1) + Ds^3(s+2)$$

$$s = 0 \Rightarrow 8 = 2A \Rightarrow A = 4$$

$$s = {}^-1 \Rightarrow 8 = {}^-D \Rightarrow D = {}^-8$$

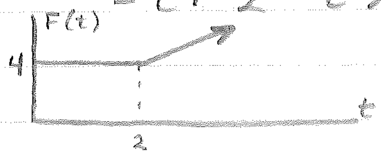
$$s = {}^-2 \Rightarrow 8 = 8C \Rightarrow C = 1$$

$$\bar{x} = \frac{4}{s^3} + \frac{1}{s+2} - \frac{8}{s+1}$$

$$x(t) = 2t^2 + e^{-2t} - 8e^{-t}$$

13) $x''(t) + x(t) = F(t)$; $x(0) = x'(0) = 0$.

$$F(t) = 4 \quad 0 < t < 2$$
$$= t + 2 \quad t > 2$$

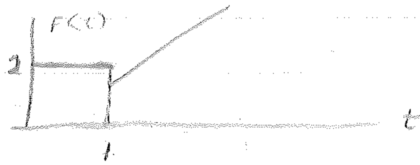


$$F(t) = 4 - 4\alpha(t-2) + (t+2)\alpha(t-2)$$
$$= 4 + [t-2]\alpha(t-2)$$
$$\mathcal{L}\{F(t)\} = \frac{4}{s} + \frac{1}{s}e^{-2s}$$
$$\therefore\ s^2\bar{x} + s\bar{x} = \frac{4}{s} + \frac{1}{s}e^{-2s}$$
$$\bar{x}(s^2+s)^0 = \frac{4}{s^2(s+1)} + \frac{e^{-2s}}{s^2(s+1)}$$
$$\bar{x}e^{2s} = \frac{5}{s^2(s+1)} \equiv \frac{A}{s^2} + \frac{B}{s+1}$$
$$5 \equiv A(s+1) + Bs^2$$
$$s = 0 \Rightarrow A = 5$$
$$s = -1 \Rightarrow 5 = B$$
$$e^{2s}\bar{x} = \frac{5}{s^2} + \frac{5}{s+1}$$
$$\mathcal{L}\left\{\frac{5}{s^2} + \frac{5}{s+1}\right\} = 5t + 5e^{t}$$
$$\mathcal{L}\{\bar{x}\} = \text{ACH!}$$

Pg 182-3

8) $F(t) = 2 \quad 0<t<1$

$\quad = t \quad t>1$



$F(t) = 2 - 2\alpha(t-1) + t\,\alpha(t-1)$

$\quad = (t-2)\alpha(t-1) + 2$

$g(t-1) = t-2$

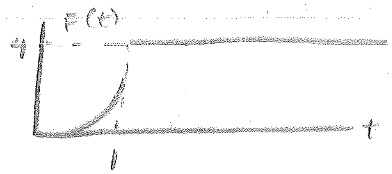
$g(t) = t-1$

$\mathcal{L}\{F(t)\} = \left(\frac{1}{s^2} - \frac{1}{s}\right)e^{-s} + \frac{2}{s}$

10) $F(t) = t^2 \quad 0<t<1$

$\quad \cancel{= 4 \quad 2<t<4}$

$\quad = 4 \quad t>1$



$F(t) = t^2 - t^2\alpha(t-1) + 4\alpha(t-1)$

$\quad = t^2 + [4-t^2]\alpha(t-1)$

$g(t-1) = 4-t^2$

$g(t) = 4-(t+1)^2$

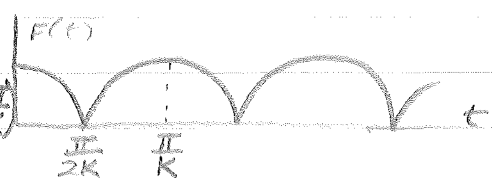
$\quad = 4 - t^2 - 2t - 1$

$\quad = 3 - t^2 - 2t$

$\mathcal{L}\{F(t)\} = e^{-s}\left(\frac{3}{s} - \frac{2}{s^3} - \frac{2}{s^2}\right) + \frac{2}{s^3}$

13) $\mathcal{L}\{|\cos kt|\} = F(s)$

$|\cos kt|_1 = \cos kt - \cos kt\,\alpha(t-\frac{\pi}{k})$



$g(t-\frac{\pi}{k}) = -\cos kt$

$g(t) = -\cos(kt+\pi)$

$\quad = \cos kt$

$\mathcal{L}\{g(t)\} = \left(\frac{s}{s^2+k^2}\right)e^{-\frac{\pi s}{k}}$

$F(s) = \frac{s}{s^2+k^2} \quad \frac{1-e^{-\frac{\pi s}{k}}}{1-e^{-\frac{\pi s}{k}}}$

Pg 186

2) $\int_0^t e^{B-t} \cos B\, dB = \Omega(t)$

$U = e^{B-t} \qquad dV = \cos B\, dB$

$dU = e^{B-t} \qquad V = \sin B$

$\Omega(t) = [e^{B-t} \sin B]_0^t + \int_0^t e^{B-t} \sin B\, dB$

$U = e^{B-t} \qquad dV = \sin B\, dB$

$dU = e^{B-t} \qquad V = \cos B$

$2\Omega(t) = \sin t + [e^{B-t} \cos B]_0^t$

$\Omega(t) = \frac{\sin t}{2} + \cancel{1} - e^{-t}\, 2^{-1}$

$\mathcal{L}\{\Omega(t)\} = \frac{1}{2}\left(\frac{1}{s^2+1} + \frac{1}{2s} - \frac{1}{s+1}\right)$

4) $\frac{K}{s(s^2+K^2)}$

$\mathcal{L}\{\frac{1}{s}\} = 1 \qquad \mathcal{L}\{\frac{K}{s^2+K^2}\} = \sin KB$

$\int_0^t \sin K(t-B)\, dB$

$\int_0^t \sin(Kt-KB)\, dB$

$\frac{1}{K}\cos(Kt-KB)\Big|_0^t$

$\frac{1}{K} - \frac{1}{K}\cos Kt$

$\frac{1}{K}(1-\cos Kt)$

Pg 192

2) $x''(t) + x(t) = 6\cos 2t$; $x(0) = 3$; $x'(0) = 1$

$s^2\bar{X} - s\,x(0) - x'0 + s\bar{X} - x(0) = \dfrac{6s}{s^2+4}$

$s^2\bar{X} - 3s - 1 + s\bar{X} - 3$

$\bar{X}(s^2+s) = \dfrac{6s}{(s^2+4)} + 3s + 4$

$\bar{X} = \dfrac{6s}{s(s+1)(s^2+4)} + \dfrac{3}{s+1} + \dfrac{4}{s(s+1)}$

$\dfrac{6}{(s+1)(s^2+4)} = \cancel{\dfrac{A}{s}} + \dfrac{B}{s+1} + \dfrac{Cs+D}{s^2+4}$

$6 = B(s^2+4) + (Cs+D)(s+1)$

$s = -1 \Rightarrow 6 = 5B \Rightarrow B = \frac{6}{5}$

$s = 2i \Rightarrow 6 = (C2i + D)(2i+1)$

$6 = -4C + 2iC + 2iD + D$

$6 = D - 4C \qquad D = -C$

$6 = ACH!$

9) [illegible] $Y = 0$

$$\sum a_n [illegible] X^{n-2} + \sum 2 a_n n X^n + \sum 5 a_n X^n = 0$$

$$\sum [illegible] + \sum a_n n(n-1) X^{n-2}$$

$$\sum a_n [illegible] X^{n-2} + \sum a_n n(n-1) X^{n-2}$$

[illegible]

$$a_2 = \frac{-a_0\,5}{2\cdot 1} \qquad a_3 = \frac{-a_1(7)}{3\cdot 2}$$

$$a_4 = \frac{-a_2\,9}{4\cdot 3} \qquad a_5 = \frac{-a_3(11)}{5\cdot 4}$$

$$a_6 = \frac{-a_4\,13}{6\cdot 5} \qquad a_7 = \frac{-a_5(15)}{7\cdot 6}$$

$$a_{2n} = \frac{a_0 a_2 \cdots a_{2n-2}\,[5\cdot 9\cdot 13 \cdots (4n+1)]}{(2n)!}$$

$$a_{2n} = \frac{a_0\,(5\cdot 9\cdot 13 \cdots (4n+1))}{(2n)!} \qquad a_{2n+1} = \frac{[illegible]\,(4n+3)}{(2n+1)!}$$

$$a_{2n+1} = \frac{a_1\,(7\cdot 11\cdot 15 \cdots (4n+3))}{(2n+1)!}$$

$$Y = \sum_{1}^{\infty} a_{2n} X^{2n} + a_0 + \sum_{1}^{\infty} a_{2n+1} X^{2n+1} + a_1 X$$

$$= \sum_{n=1}^{\infty} \frac{a_0(5\cdot 9\cdot 13 \cdots (4n+1))}{(2n)!} X^{2n} + \sum_{n=1}^{\infty} \frac{a_1(7\cdot 11\cdot 15 \cdots (4n+3))}{(2n+1)!} X^{2n+1}$$

$$+ a_0 + a_1 X$$

4. 289

1. $(x^2+4)y'' + 2xy' - 12y = 0$

$x^2y'' + 4y'' + 2xy' - 12y = 0$

$\sum a_n n(n-1)x^n + \sum_2 4a_n n(n-1)x^{n-2} + \sum 2a_n n x^n - \sum_0 12x^n a_n = 0$

$\sum_0 x^n a_n\left[n(n-1) + 2n - 12\right] + \sum_2 4a_n n(n-1)x^{n-2}$

$\sum_2 x^{n-2} a_{n-2}\left[(n-2)(n-3) + 2(n-2) - 12\right] + \sum_2 4a_n n(n-1)x^{n-2}$

$\sum_2 x^{n-2} a_{n-2}\left[n^2 - 3n - 10\right] + \sum_2 4a_n n(n-1)x^{n-2}$

$(n-5)(n+2)$

$a_{n-2}(n-5)(n+2) = -4a_n n(n-1)$

$$a_n = -\frac{a_{n-2}(n-5)(n+2)}{4n(n-1)}$$

$a_3 = -\frac{a_1(-2)(5)}{4 \cdot 3 \cdot 2}$ $\qquad a_2 = -\frac{a_0(-3)(4)}{4 \cdot 2 \cdot 1}$

$a_5 =$ $\qquad a_4 = -\frac{a_2(-1)(6)}{4 \cdot 4 \cdot 3}$

$a_6 = -\frac{a_4 \cdot 1 \cdot 8}{4 \cdot 6 \cdot 5}$

$a_2, a_4, \ldots a_{2n} = \frac{(-1)^n a_0 (-3 \cdot -1 \cdot 1 \cdots (2n-5))(4 \cdot 6 \cdot 8 \cdots 2n+2)}{4^n (2n)!}$

etc.

Pg 314

2) $4xy'' + 3y' - 3y = 0$

$$\sum_{n=0}^{\infty} 4(n+c)(n+c-1)x^{n+c-1} a_n + \sum_{0}^{\infty} 3a_n(n+c)x^{n+c-1} - \sum_{0}^{\infty} 3a_n x^{n+c} = 0$$

$$\sum_{n=0}^{\infty} a_n x^{n+c-1}\left[(n+c)(4n+4c-1)\right] - \sum_{n=0}^{\infty} 3a_n x^{n+c} = 0$$

$$\sum_{n=0}^{\infty} a_n x^{n+c-1}\left[(n+c)(4n+4c-1)\right] - \sum_{1}^{\infty} 3a_{n-1} x^{n+c-1} = 0$$

At $n=0$

$c(4c+1)=0$

$c_1 = 0 \; ; \; c_2 = \frac{1}{4}$

For $n \geq 1$

$a_n (n+c)(4n+4c-1) = 3a_{n-1}$

$$a_n = \frac{3a_{n-1}}{(n+c)(4n+4c-1)}$$

$c_1 = 0$

$$a_n = \frac{3a_{n-1}}{n(4n-1)} \; ; \; n \geq 1$$

$$a_1 = \frac{3a_0}{1 \cdot 3}$$

$$a_2 = \frac{3a_1}{2 \cdot 7}$$

$$a_3 = \frac{3a_2}{3 \cdot 11}$$

$$a_1 \cdot a_2 \cdots a_n = \frac{3^n a_0 a_1 \cdots a_{n-1}}{3!\,(3 \cdot 7 \cdot 11 \cdots (4n-1))}$$

$$a_n = \frac{3^n a_0}{3!\,(3 \cdot 7 \cdot 11 \cdots (4n-1))}$$

$$y_1 = 1 + \sum_{n=1}^{\infty} a_n x^{n+c}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{3^n x^n}{3!\,(3 \cdot 7 \cdot 11 \cdots (4n-1))}$$

714

4) $2xy'' + 5(1+2x)y' + 5y = 0$

$$\sum_{0}^{\infty} 2(n+c)(n+c-1)a_n x^{n+c-1} + 5\sum_{0}^{\infty}(n+c)a_n x^{n+c-1} + \sum_{0}^{\infty} 10(n+c)a_n x^{n+c} + \sum_{0}^{\infty} 5a_n x^{n+c} = 0$$

$$\sum_{0}^{\infty} a_n x^{n+c-1}(n+c)\left[(2n+2c+3)\right] + \sum_{0}^{\infty} a_n x^{n+c}(10n+10c+5) = 0$$

$$\sum_{0}^{\infty} a_n x^{n+c-1}(n+c)(2n+2c+3) + \sum_{1}^{\infty} a_{n-1} x^{n+c-1}(10n+10c-5) = 0$$

$$a_n = \frac{-5(2n+2c-1)a_{n-1}}{(n+c)(2n+2c+3)} \quad ; n \geq 1$$

$$a_0 c(2c+3) = 0 \quad ; n = 0$$

$$c_1 = 0 \; ; \; c_2 = -\frac{3}{2}$$

$c = 0$;

$$a_n = \frac{-5(2n-1)a_{n-1}}{n(2n+3)}$$

$$a_1 = \frac{-5 \cdot 1 \cdot a_0}{1 \cdot 5}$$

$$a_2 = \frac{-5(3)a_1}{2 \cdot 7}$$

$$a_3 = \frac{-5 \cdot 5 \cdot a_2}{3 \cdot 9}$$

$$a_4 = \frac{-5 \cdot 7 \cdot a_3}{4 \cdot 11}$$

$\vdots$

$$a_n = \frac{(-5)^n\left[1 \cdot 3 \cdot \cancel{5} \cdot \ldots \cdot (2n-1)\right] a_0}{n!\left[\cancel{5} \cdot \cancel{7} \cdot 11 \cdot \ldots \cdot (2n+3)\right]}$$

$$= \frac{(-5)^n (3) a_0}{n!(2n+1)(2n+3)} \quad n \geq 1 \; ; \quad Y_1 = 1 + \sum_{n=1}^{\infty} \frac{(-5)^n 3 a_0 x^n}{n!(2n+1)(2n+3)}$$

$c = -\frac{3}{2}$

$$a_n = \frac{-5(2n-4)a_{n-1}}{\left(n-\frac{3}{2}\right)(2n)} = \frac{-20(n-4)a_{n-1}}{2(2n-3)n}$$

$$a_1 = \frac{-20 \cdot 4 a_0}{-2}$$

$$a_2 = \frac{-20 \cdot 2 \, a_1}{2 \cdot 4 \cdot 2} =$$

$\vdots \qquad \vdots$

3) $2X^2Y'' - 2X^2Y' - XY' + Y - 5XY = 0$

$$\sum_0^\infty 2a_n(n+c)(n+c-1)X^{n+c} - \sum_0^\infty 2a_n(n+c)X^{n+c+1} - \sum_0^\infty a_n(n+c)X^{n+c} + \sum_0^\infty a_nX^{n+c} - \sum_0^\infty 5a_nX^{n+c+1}$$

$$\sum_0^\infty a_nX^{n+c}\left[2(n+c)(n+c-1) - (n+c) + 1\right] - \sum_0^\infty a_nX^{n+c+1}\left[2(n+c)+5\right] = 0$$

$$\sum_0^\infty a_nX^{n+c}\left[(n+c)(2n+2c-3)+1\right] - \sum_0^\infty a_nX^{n+c+1}(2n+2c+5) = 0$$

$$-\sum_1^\infty a_{n-1}X^{n+c}(2n+2c+3)$$

AT $n=0$

$c(2c-3)+1=0$

$2c^2-3c+1=0$

$(2c-1)(c-1)=0$

$c_1=\frac{1}{2};\ c_2=1$

AT $n\geq 1$

$a_n[(n+c)(2n+2c-3)+1] = a_{n-1}(2n+2c+3)$

$$a_n = \frac{a_{n-1}(2n+2c+3)}{(n+c)(2n+2c-3)+1};\ n\geq 1$$

FOR $c=1$;

$$a_n = \frac{a_{n-1}(2n+5)}{(n+1)(2n-1)+1} = \frac{a_{n-1}(2n+5)}{2n^2+n} = \frac{a_{n-1}(2n+5)}{n(2n+1)};\ n\geq 1$$

$$a_1 = \frac{a_0 8}{2\cdot 1+1} = \frac{a_0 7}{3} \qquad a_1 = \frac{a_0(7)}{1\cdot 3}$$

$$a_2 = \frac{a_1(4)}{3\cdot 3+1} = \frac{a_1 9}{10} \qquad a_2 = \frac{a_1 9}{[illegible]}$$

$$a_3 = \frac{a_2 11}{4\cdot 7+1} = \frac{a_2 11}{29} \qquad a_3 = \frac{a_2 11}{3\cdot 7}$$

$$a_4 = \frac{a_3 13}{5\cdot 7+1} = \frac{a_3 13}{36} \qquad a_4 = \frac{a_3 13}{[illegible]}$$

$$a_1 a_2 a_3 \cdots a_k = \frac{(a_0 a_1 a_2 \cdots a_{k-1})(7\cdot 9\cdot 11\cdot 13 \cdots (2k+5))}{k!\,(3\cdot 5\cdot 7\cdot 9\cdots(2k+1))}$$

$$a_n = \frac{a_0(2n+3)(2n+5)}{n!\,15}$$

$$\therefore Y_1 = X + \sum_{n=1}^\infty \frac{(2n+3)(2n+5)}{15\,n!}X^{n+1}$$

$$a_n = \frac{a_{n-1}(2n+2c+3)}{(n+c)(2n+2c-3)-1}$$

For $c = \frac{1}{2}$

$$a_n = \frac{a_{n-1}(2n+4)}{(n+\frac{1}{2})(2n-2)-1}$$

$$= \frac{2a_{n-1}(n+2)}{(2n+1)(n-1)-1}$$

$$= \frac{2a_{n-1}(n+2)}{2n^2-n-2}$$

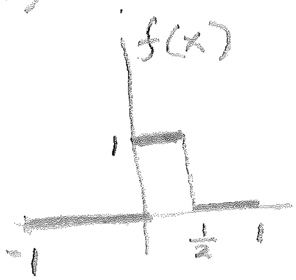
$$a_n = \frac{2a_{n-1}(n+2)}{(2n+1)(n-2)}, \quad n \geq 1$$

$$a_1 = \frac{-2a_0 3}{3-1}$$

$a_2 =$

Pg. 394)

10)



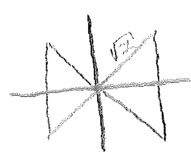
$$a_n = \int_{-1}^{1} f(x)\cos\frac{n\pi x}{c}\,dx$$
$$= \int_0^{\frac{1}{2}} \cos\frac{n\pi x}{c}\,dx$$
$$= \left[\sin\frac{n\pi x}{c}\right]_0^{\frac{1}{2}} \frac{c}{n\pi}$$
$$= \frac{c}{n\pi}\sin\frac{n\pi}{2c} = \frac{1}{n\pi}\sin\frac{n\pi}{2}$$

$$b_n = \int_0^{\frac{1}{2}} \sin\frac{n\pi x}{c}\,dx$$
$$= \left[\frac{c}{n\pi}\cos\frac{n\pi x}{c}\right]_{\frac{1}{2}}^{0}$$
$$= \frac{c}{n\pi}\left(1-\cos\frac{n\pi}{2c}\right)$$
$$= \frac{1}{n\pi}\left(1-\cos\frac{n\pi}{2}\right)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{c} + b_n\sin\frac{n\pi x}{c}\right)$$

$$a_0 = \int_0^{\frac{1}{2}} dx = [x]_0^{\frac{1}{2}} = \frac{1}{2}$$

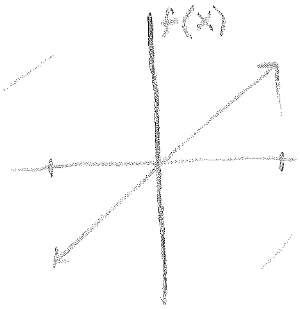
$$f(x) = \frac{1}{4}\ \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{\sin\frac{n\pi}{2}}{n}\cos n\pi x + \frac{\left(1-\cos\frac{n\pi}{2}\right)}{n}\sin n\pi x\right]$$



Pg 394

2) $^-c < x < c$

$f(x) = x$



$$a_n = \frac{1}{c}\int_{-c}^{c} x\cos\frac{n\pi x}{c}\,dx$$

$$v = \frac{x}{c} \qquad du = \cos\frac{n\pi x}{c}\,dx$$

$$dv = \frac{1}{c}\,dx \qquad u = \frac{c}{n\pi}\sin\frac{n\pi x}{c}$$

$$a_n = \left[\frac{x}{n\pi}\sin\frac{n\pi x}{c}\right]_{-c}^{c} - \int_{-c}^{c}\frac{1}{n\pi}\sin\frac{n\pi x}{c}$$

$$= \left[\frac{x}{n\pi}\sin\frac{n\pi x}{c}\right]_{-c}^{c} + \left[\frac{c}{n^2\pi^2}\cos\frac{n\pi x}{c}\right]_{-c}^{c}$$

$$= \frac{c}{n\pi}\sin {}^+n\pi + \frac{{}^+c}{n\pi}\sin {}^-nx + \frac{c}{n^2\pi^2}\cos n\pi$$

$$- \frac{c}{n^2\pi^2}\cos {}^+n\pi$$

$$= \cos\pi n\left(\frac{c}{n^2\pi^2} - \frac{c}{n^2\pi^2}\right) = 0$$

$$b_n = \frac{1}{c}\int_{-c}^{c} x\sin\frac{n\pi x}{c}\,dx$$

$$v = \frac{x}{c} \qquad du = \sin\frac{n\pi x}{c}\,dx$$

$$dx = \frac{dx}{c} \qquad u = {}^-\frac{c}{n\pi}\cos\frac{n\pi x}{c}$$

$$b_n = \left[\frac{{}^-x}{n\pi}\cos\frac{n\pi x}{c}\right]_{-c}^{c} + \int_{-c}^{c}\frac{{}^+1}{n\pi}\cos\frac{n\pi x}{c}\,dx$$

$$= \frac{{}^-c}{n\pi}\cos n\pi - \frac{c}{n\pi}\cos {}^+n\pi + \left[\frac{c}{(n\pi)^2}\sin\frac{n\pi x}{c}\right]_{-c}^{c}$$

$$= {}^-\frac{2c}{n\pi}\cos n\pi + \frac{c}{(n\pi)^2}\sin n\pi + \frac{c}{(n\pi)^2}\sin {}^+n\pi$$

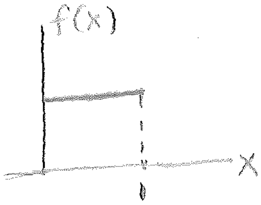
$$= {}^-\frac{2c}{n\pi}\cos n\pi$$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{c} + b_n\sin\frac{n\pi x}{c}\right)$$

$$= \sum_{n=1}^{\infty}\frac{2c}{n\pi}(-1)^{n+1}\sin\frac{n\pi x}{c}$$

pg 398)

1)



$$f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c} \;;\; 0<x<c$$

$$c=1$$

$$f(x)=\sum_{n=1}^{\infty} b_n \sin n\pi x$$

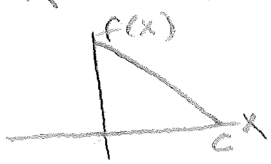
$$b_n=\frac{2}{c}\int_0^c \sin\frac{n\pi x}{1}\,dx$$

$$=\frac{-2}{n\pi}\Big[\cos n\pi x\Big]_0^1$$

$$=\frac{2}{n\pi}\left(1-\cos n\pi\right)$$

$$\therefore f(x)=\sum_{n=1}^{\infty}\frac{2\{1-(-1)^{n+1}\}}{n\pi}\sin n\pi x$$

4) $0<x<c$ $\quad f(x)=c-x$



$$b_n=\frac{2}{c}\int_0^c \sin\frac{n\pi x}{c}(c-x)$$

$$=\frac{2}{c}\int_0^c \sin\frac{n\pi x}{c}-\frac{2}{c}\int_0^c x\sin\frac{n\pi x}{c}$$

$$=\left[\frac{2}{cn\pi}\cos\frac{n\pi x}{c}\right]_c^0+\left[\frac{2x\cancel{c}}{\cancel{c}n\pi}\cos\frac{n\pi x}{c}-\frac{2c}{n^2\pi^2}\sin\frac{n\pi x}{c}\right]_0^c$$

$$b^n=\left(\frac{2}{cn\pi}-\frac{2}{cn\pi}\cos n\pi\right)+\frac{2c}{n\pi}\cos n\pi-\cancel{\frac{2c}{n^2\pi^2}\sin}\,n\pi+0$$

$$=\frac{2}{cn\pi}(1-\cos n\pi)+\frac{2c}{n\pi}\cos n\pi$$

$$=\frac{2}{cn\pi}\left[1-(-1)^{n+1}\right]+\frac{2c}{n\pi}(-1)^{n+1}$$

$f(x)=$ ~

[illegible] of the following functions are periodic [illegible] the fundamental (lowest) periods:

[illegible] d) $\cos(\pi x/a)$ [illegible]

[illegible] e) $\tan \pi x$

[illegible] f) [illegible]

[illegible] whether the following functions are even, odd, or neither.

[illegible] e) $\cos(\pi x + 1)$

[illegible] f) $\tan \pi x$

[illegible] g) $\dfrac{\sin x}{x}$

[illegible] h) $x^2 + x + 1$

[illegible] the function $f(x) = x(1-x)$, $0 \le x \le 1$.

a) Find a continuation of this function in the interval $-1 \le x \le 0$ such that the continued function is even in $-1 \le x \le 1$. If, moreover, $f(x) = f(x+2)$, sketch the function for $-3 \le x \le 3$. What is the fundamental period?

b) Find a continuation of $f(x) = x(1-x)$, $0 \le x \le 1$ in the interval $-1 \le x \le 0$ such that the continued function is odd in $-1 \le x \le 1$. If, moreover, $f(x) = f(x+2)$, sketch the function for $-3 \le x \le 3$. What is the fundamental period?

[illegible] continuation of $f(x) = x(1-x)$, $0 \le x \le 1$ in the interval [illegible] the continued function is neither even or odd [illegible] [illegible], then $f(x) = f(x+2)$; sketch the function for $-3 \le x \le 3$.

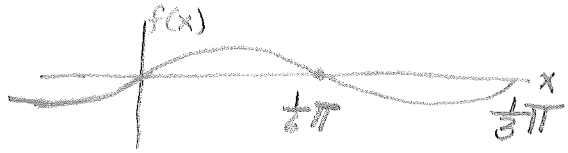
[illegible] $f(x)$ [illegible] if $f(x) = f(x+1)$ in [illegible]

[illegible] expansions for the functions [illegible]

[illegible]

PROB ON FOUR. SERIES

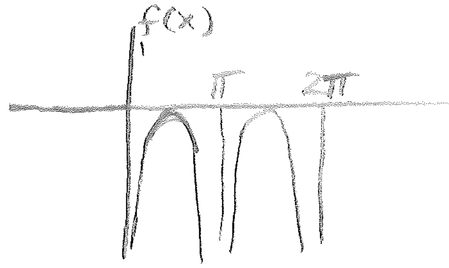
1) a) $\sin 6x$



$\Rightarrow$ PERIODIC; $T = \frac{\pi}{3}$

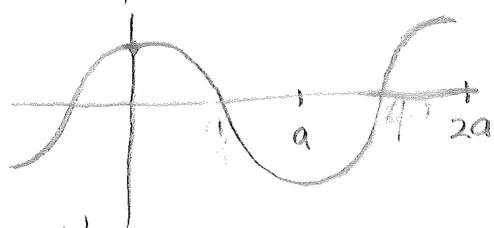
b) $x^2 \rightarrow$ NOT PERIODIC

c) $\ln|\sin x|$



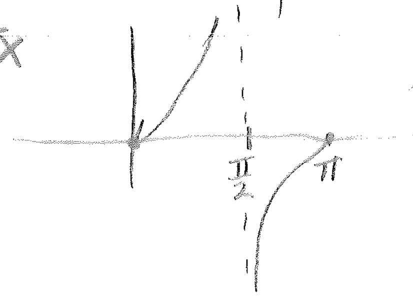
x $\Rightarrow$ PERIODIC; $T = \pi$

d) $\cos\left(\frac{\pi x}{a}\right)$



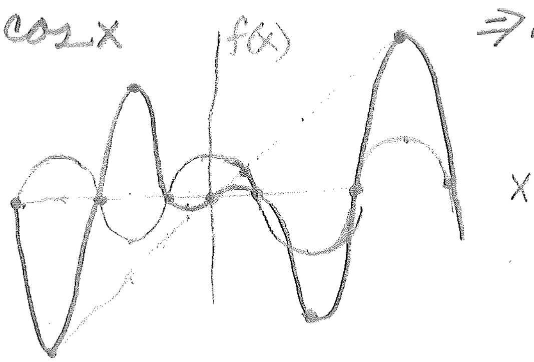
$\Rightarrow$ PERIODIC; $T = 2a$

e) $\tan \pi x$



$\Rightarrow$ PERIODIC; $T = \pi$

f) $x \cos x$



$\Rightarrow$ NOT PERIODIC

PROBLEMS FOR EXTRA CREDIT

1 Solve $y'' + y = \sec t$, $y(0) = 0$, $y'(0) = 0$ and obtain the solution in terms of elementary functions.

2 Evaluate $\int_0^\infty t e^{-3t} \sin 2t \, dt$ using the idea of the Laplace Transform

3 By taking Laplace transforms and transforming back, evaluate the integrals

$$F(t) = \int_0^\infty \frac{\sin tx}{x} \, dx \qquad t > 0$$

$$F(t) = \int_0^\infty \frac{\cos tx}{1 + x^2} \, dx \qquad t > 0$$

4. Prove that $\mathcal{L}\{t^{-1} F(t)\} = \int_s^\infty f(u) \, du$ if $\mathcal{L}\{F(t)\} = f(s)$.
(Hint: use the rules $\mathcal{L}\{t F(t)\} = -f'(s)$ and $\lim_{s \to \infty} f(s) = 0$)
Show that

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \arctan \frac{1}{s}$$

$$\mathcal{L}\left\{\frac{1 - e^{-t}}{t}\right\} = \ln\left(1 + \frac{1}{s}\right)$$

5 Solve the following problems from § 71, p 225-226: 1, 3, 5, 7, 11

6 If $\mathcal{L}\{F(t)\} = \frac{1}{s^2} - \frac{2}{s} \frac{e^{-s}}{1 - e^{-2s}}$, obtain $F(t)$ in the intervals $0 < t < 1$, $1 < t < 2$, $2 < t < 3$ and $3 < t < 4$. Show that $F(t)$ is a periodic function and determine its period
(Hint: expand $\frac{1}{1 - e^{-2s}}$ in powers of e^{-2s})

Marks

385 - 2

DE 2 Take-home Test $\frac{47}{60}$ C+

To be handed in Thursday Jan, 22 at 9.40. (If you have already two other tests for Thursday I am willing to grant an extension provided you ask for it before the deadline.)
you may consult Thomas

1 a) Derive a_k in the power series expansion $\cos x = \sum_{k=0}^{\infty} a_k x^k$

$f_k(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$

$a_0 = f(0)$ $\quad f(x) = \cos x \rightarrow f(0) = 1$

$a_1 = f'(0)$ $\quad f'(x) = -\sin x \rightarrow f'(0) = 0$

$a_2 = \frac{f''(0)}{2!}$ $\quad f''(x) = -\cos x \rightarrow f''(0) = -1$

$\vdots$

$a_n = \frac{f^n(0)}{n!}$ $\quad f'''(x) = \sin x \rightarrow f'''(0) = 0$

(REPEATS) (REPEATS)

3/5

$a_k = \frac{(-1)^n}{(2n)!}$

$\sum_0^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$

$$\therefore \cos x = 1 + 0 - \frac{x^2}{2!} + 0 - \frac{x^4}{4!} + \ldots + \frac{x^{2n}}{(2n)!} = \sum_{m=0}^{\infty} \frac{x^{2m}(-1)^m}{(2m)!}$$

$K = 2M$

$$\left(\cos x = \sum_{k=0}^{\infty} \frac{x^k (-1)^{\frac{k}{2}}}{k!} \Rightarrow a_k = \frac{(-1)^{k/2}}{k!} \right)^*$$

$(-1)^{\frac{1}{2}} = ?$

* a_k MUST BE REAL ie $(-1)^{k/2}$ MUST BE REAL. ALL IMAGINARY a_k (FOR ODD K) CAN THUS BE SET TO 0. ONLY REAL a_k (FOR EVEN K) SHOULD BE INCLUDED IN THE SUMMATION, FOR COS X IS REAL IF X IS REAL.

why

b) Find the terms up to degree 3 in the Taylor series expansion of $\tan x$ around the point $a = \frac{1}{4}\pi$

TAYLOR SERIES:

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots + \frac{(x-a)^n}{n!} f^n(a)$$
$$= \sum_{k=0}^{\infty} \frac{(x-a)^k}{k!} f^k(a)$$

$f(x) = \tan x \; ; \; a = \frac{\pi}{4}$

4/5

$f(x) = \tan x \longrightarrow f(\frac{\pi}{4}) = 1$

$f'(x) = \sec^2 x \longrightarrow f'(\frac{\pi}{4}) = 2$

$f''(x) = 2\sec^2 x \tan x \rightarrow f''(\frac{\pi}{4}) = 4$

$f'''(x) = 2\sec^2 x \tan^2 x + 2\sec^4 x \rightarrow f'''(\frac{\pi}{4}) = 6$ 16

$$\tan(x) \cong 1 + 2(x - \frac{\pi}{4}) + 4 \frac{(x-\frac{\pi}{4})^2}{2!} + 6 \frac{(x-\frac{\pi}{4})^3}{3!}$$

$$\tan x \cong 1 + 2(x - \frac{\pi}{4}) + 2(x - \frac{\pi}{4})^2 + \frac{8}{3}(x - \frac{\pi}{4})^3$$

7/10

The quantity $\sqrt{e} = e^{0.5}$ is to be computed from the series $e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + R_n(x, 0)$

Show that we have to take $n \geq 4$ in order to have an error smaller then .0005. Compute $\sqrt{e}$ to 3D accuracy

a) $R_n(x,0) = \int_0^x \frac{(x-t)^n}{n!} f^{n+1}(t)\,dt$

$x = .5$

$R_n(.5,0) = \int_0^{.5} \frac{(.5-t)^n}{n!} f^{n+1}(t)\,dt$

$0 \leq t \leq .5 (=x)$

$e^{t} \leq e^{.5(=x)} < 3^{.5(=x)}$. $f^{n}(e^{x}) = e^{x}$

$\therefore R_n(.5,0) < 3^{.5} \int_0^{.5} \frac{(.5-t)^n}{n!}\,dt = \sqrt{3}\,\frac{(.5)^{n+1}}{(n+1)!}$

$R_n < 5 \times 10^{-4}$

$\rightarrow 5 \times 10^{-4} > \sqrt{3}\,\frac{(.5)^{n+1}}{(n+1)!}$

$2.8 \times 10^{-4} > \frac{.5^{n+1}}{(n+1)!}$

TRY $n = 3$

$\frac{.5^4}{4!} = 2.6 \times 10^{-3} \not< 2.8 \times 10^{-4}$

TRY $n = 4$

$\frac{.5^5}{5!} = 2.61 \times 10^{-4} < 2.8 \times 10^{-4}$

$\therefore n = 4$ TO INSURE ACCURACY TO 3 DECIMALS

b) $n = 1 \Rightarrow e_1^{.5} = 1.000$

$n = 2 \Rightarrow e_2^{.5} = e_1^{.5} + .5 = 1.500$

$n = 3 \Rightarrow e_3^{.5} = e_2^{.5} + .125 = 1.625$

$n = 4 \Rightarrow e_4^{.5} = e_3^{.5} + .028 = 1.653$

$\sqrt{e} = 1.648$

10/10

Show that the ordinate y of the catenary

$$y = \cosh x = \tfrac{1}{2}(e^{x} + e^{-x})$$

deviates by less than .003 from the ordinate of the parabola $x^2 = 2(y-1)$

over the range $|x| \le \frac{1}{3}$

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = \underset{0}{1} + \underset{1}{\frac{x^2}{2!}} + R_3(x,0)$$

$$Y_A = \frac{x^2}{2} + 1$$

$\therefore$ SINCE $\cosh x = Y_A + R_3(x,0)$, IT MUST BE SHOWN THAT $R_3(x,0) < 3 \times 10^{-3}$ FOR $|x| \le \frac{1}{3}$

$$R_3(x,0) = \int_0^x \frac{(x-t)^3}{6} f^{iv}(t)\,dt$$

$$0 < t \le x \le \tfrac{1}{3} \quad \cosh x \le 1 \longleftarrow (|x| \le \tfrac{1}{3})$$

$$R_3(x,0) \le \frac{1}{6}\int_0^x (x-t)^3\,dt$$

Let $x = \frac{1}{3}$

$$R_3(x,0) \le \frac{1}{6}\int_0^{1/3} \left(\tfrac{1}{3} - t\right)^3 dt$$

$$\le \frac{1}{24}\left[\left(\tfrac{1}{3} - t\right)^4\right]_{\frac{1}{3}}^{0}$$

$$\le .000514$$

$\therefore R_3(x,0) < .003$ FOR POINTS FROM 0 TO $\frac{1}{3}$, AND THUS FOR POINTS FROM $-\frac{1}{3}$ TO 0

9/10

4. By expanding numerator and denominator in MacLaurin series determine

$$\lim_{x \to 0} \frac{e^x - 1}{\ln(1+x)}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$$
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots + \frac{x^n}{n}$$

$$Y = \lim_{x \to 0} \frac{e^x - 1}{\ln(x+1)}$$
$$= \lim_{x \to 0} \frac{x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}}{x - \frac{x^2}{2} + \frac{x^3}{3} + \dots + \frac{x^n}{n!}}$$
$$= \lim_{x \to 0} \frac{1 + \frac{x}{2!} + \frac{x^2}{3!} + \dots + \frac{x^{n-1}}{n!}}{1 + \frac{x}{2} + \frac{x^2}{3} + \dots + \frac{x^{n-1}}{n}}$$
$$= 1$$

5

b) Determine a such that $\lim_{x \to 0} x^{-5}(\arctan x - \frac{1}{a}\sin ax)$ is finite and evaluate that limit

$$Y = \frac{\tan^{-1} x - \frac{1}{a}\sin ax}{x^5}$$
$$Y = \frac{\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\right) - \frac{1}{a}\left(ax - \frac{a^3x^3}{3!} + \frac{a^5x^5}{5!} - \frac{a^7x^7}{7!} + \dots\right)}{x^5}$$
$$Y = \frac{\left(\cancel{x} - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\right) - \left(\cancel{x} - \frac{a^2x^3}{3!} + \frac{a^4x^5}{5!} - \frac{a^6x^7}{7!} + \frac{a^8x^9}{9!} \dots\right)}{x^5}$$
$$Y = \left(\frac{-1}{3x^2} + \frac{1}{5} - \frac{x^2}{7} + \frac{x^4}{9} + \dots\right) - \left(\frac{-a^2}{x^2 3!} + \frac{a^4}{5!} - \frac{a^6x^2}{7!} + \frac{a^8x^4}{9!} \dots\right)$$
$$\lim_{x \to 0} Y = \frac{-1}{3x^2} + \frac{1}{5} + \frac{a^2}{6x^2} - \frac{a^4}{120}$$

FOR $\lim_{x \to 0} Y \neq \infty$, AND THUS BE FINITE, $\frac{-1}{3x^2} + \frac{a^2}{6x^2} = 0$

$$\frac{1}{3x^2} = \frac{a^2}{6x^2}$$
$$a = \sqrt{2}$$

5 THUS $\lim_{x \to 0} Y = \frac{1}{5} - \frac{a^4}{120} = \frac{1}{5} - \frac{(\sqrt{2})^4}{120} = \frac{5}{30} = \frac{1}{6}$

$\frac{10}{10}$

2. Find the interval of convergence for the series:

(i) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n}$

$S^* = \sum_{n=0}^{\infty} \frac{|x|^{2n}}{4^n (n!)^2}$

RATIO TEST FOR CONVERGENCE

$\frac{a_{n+1}}{a_n} < 1$

$\frac{|x|^{2n+2}}{4^{n+1}[(n+1)!]^2} \cdot \frac{4^n\, n!}{|x|^{2n}}$

$= \frac{|x|^2}{4(n+1)^2}$

$\lim_{n\to\infty} \frac{|x|^2}{4(n+1)^2} = 0 < 1$

$\therefore$ THIS SERIES IS CONVERGENT FOR $-\infty < x < \infty$, BY THEOROM 8

4

(ii) $\sum_{n=0}^{\infty} \frac{(x+2)^n}{3^n\, n}$

RATIO TEST FOR CONVERGENCE

$\frac{a_{n+1}}{a_n} = \frac{(x+2)^{n+1}}{3^{n+1}(n+1)} \cdot \frac{3^n\, n}{(x+2)^n} = \frac{(x+2)n}{3(n+1)}$

$= \frac{x+2}{3+3/n}$

$\lim_{n\to\infty} \frac{x+2}{3+3/n} = \left(\frac{x+2}{3} \left(< 1\ (?)\right)\right)$

$|x+2| < 3$

$x = 1$

$\therefore$ x IS DIVERGENT FOR VALUES $\neq 1$

TEST FOR CONVERGENCE AT $x = 1$

$\sum_{n=0}^{\infty} \frac{3^n}{3^n n} = \sum_{n=0}^{\infty} \frac{1}{n}$ WHICH IS A HARMONIC SERIES, WHICH IS DIVERGENT.

$\therefore \sum_{n=0}^{\infty} \frac{(x+2)^n}{3^n n}$ IS DIVERGENT FOR ALL X

$\frac{4}{20}$

4 Determine whether the following alternating series are convergent or divergent:

(i) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n+1}{1000n} = \frac{2}{1000} + \frac{3}{2000} - \frac{4}{3000} + \ldots$

$\lim_{n\to\infty} \frac{n+1}{1000n} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1000} = \frac{1}{1000}$

$\therefore$ THE SERIES DIVERGES, FOR AS n APROACHES INFINITY, THE nth TERM DOES NOT APROACH 0.

4

(ii) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n}{n^2+9} = \frac{2}{10} - \frac{4}{13} + \frac{9}{18} - \frac{10}{34} + \ldots$

1) THE SERIES IS STRICTLY ALTERNATING

2) $\lim_{n\to\infty} \frac{2n}{n^2+9} = \lim_{n\to\infty} \frac{2}{n+9/n} = 0$

3

AS n APROACHES INFINITY, THE nth TERM APROACHES 0.

3) $\frac{a_{n+1}}{a_n} = \frac{(2n+2)(n^2+9)}{[(n+1)^2+9]2n} = \frac{2n^3+2n^2+18n+18}{2n^3+4n^2+10n}$

$= \frac{2+2/n+18/n^2+18/n^3}{2+4/n+10/n^2}$

also $\cancel{\frac{a_{n+1}}{a_n}}$ $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 1$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 \Rightarrow a_{n+1} < a_n \ \forall n$?

EACH TERM IS NUMERICALLY LESS THAN OR EQUAL TO, ITS PREDECESSOR

THE THREE CONDITIONS TO HAVE CONVERGENCE IN AN ALTERNATING SERIES HAVE BEEN MET. THIS SERIES IS CONVERGENT

7/10

25/80

F

DE II Quiz ?-?-1970

Name ROBERT J. MARKS

Box 385-2

1. Find the Laplace transform of the functions:

a. $t^2+3t-5 = \int_0^\infty e^{-st}(t^2-3t-5)\,dt = \left[\frac{-e^{-st}}{s}(t^2-3t-5)\right]_0^\infty + \int_0^\infty \frac{e^{-st}}{s}(2t-3)\,dt$ (SEE BACK)

$dv = e^{-st}dt \quad u = t^2-3t-5$

$v = \frac{-e^{-st}}{s} \quad du = 2t-3$

$= \frac{2}{s^3} - \frac{3}{s^2} - \frac{5}{s}$

Use sheet

b. $2e^{3t} - e^{-2t}$

$\mathcal{L}\{2e^{3t} - e^{-2t}\} = \frac{2}{s-3} - \frac{1}{s+2}$

c. $e^{-2t}\sin 3t$

8/20

d. $\sinh kt \sin kt = \frac{1}{2}\int_0^\infty e^{-st}(e^{kt} - e^{-kt}) \sin kt\, dt$

e. $t^2 \sin kt$ - OVER

2. Find the inverse transform of the functions:

a. $\frac{3s+4}{s^2} = \frac{3s}{s^2} + \frac{4}{s^2} = \frac{3}{s} + \frac{4}{s^2} = \mathcal{L}\{3+4t\}$

8/20

b. $\frac{s+2}{s^2-6s+8} = \frac{s+2}{(s-3)^2+\cancel{7}} \Rightarrow e^{3t}\cancel{\mathcal{L}\left\{\frac{s+5}{s^2+2}\right\}} = e^{3t}\mathcal{L}\left\{\frac{s}{s^2+2} + \frac{5}{s^2+2}\right\}$

$e^{3t}\left(\cosh\sqrt{2}t - \frac{\sin\sqrt{2}t}{\sqrt{2}}\right)$

c. $\frac{s^2}{(s+2)^3} \Rightarrow s^2 \equiv A(s+2) + B(s+2)^2$

$s=-1 \Rightarrow 1 = A+B$

$s=1 \Rightarrow 1 = 3A+9B \Rightarrow B = -\frac{1}{6},\ A = \frac{5}{6}$

$\frac{1}{6(s+2)} + \frac{5}{6(s+2)^2}$

$= \cancel{\mathcal{L}\left\{\frac{\sin\sqrt{2}t}{6t} + \frac{5}{12}(\sin\sqrt{2}t + \sqrt{5}t\ldots)\right\}}$

d. $\frac{s}{s^2-6s+13} = \frac{s}{(s-3)^2+4} \Rightarrow e^{3t}\mathcal{L}\left\{\frac{s+3}{s^2+4}\right\} = e^{-3t}\mathcal{L}\left\{\frac{s}{s^2+4} + \frac{3}{s^2+4}\right\}$

$e^{+3t}\left(\cos 2t + \frac{3}{2}\sin 2t\right)$

e. $\frac{e^{-2s}}{(s-2)(s-1)} =$

$\frac{1}{(s-2)(s-1)} = \frac{A}{s-2} + \frac{B}{s-1} = \frac{1}{s-2} - \frac{1}{s-1} = \mathcal{L}^{-1}\{e^{2t} - e^{t}\}$

$1 \equiv A(s-1) + B(s-2)$

$s=1 \Rightarrow B = -1$

$s=2 \Rightarrow A = 1$

$\left[e^{2(t-2)} - e^{(t-2)}\right]\xi(t-2)$

16

1) $\int_0^\infty \frac{e^{-st}}{s}(2t-3)\,dt$

$dV = \frac{e^{-st}}{s}$ $\quad U = 2t-3$

$V = \frac{-e^{-st}}{s^2}$ $\quad dU = 2dt$

$$\mathcal{L}(t^2-3t+5) = \left[\frac{-e^{-st}}{s}(t^2-3t-5)\right]_0^\infty - \left[\frac{e^{-st}}{s^2}(2t-3)\right]_0^\infty + \left(\frac{2}{s^2}\int_0^\infty e^{-st}\right)$$

$$\left[\frac{2e^{-st}}{-s^3}\right]_0^\infty$$

$$\mathcal{L}\{t^2-3t-5) = \frac{-5}{s} - \left(\frac{+3}{s^2}\right) + \left(+\frac{2}{s^3}\right)$$

3) $e^{-2t}\sin 3t$

$dV = \frac{e^{-2t}}{t}$ $\quad U = 3\cos 3t$

$$\mathcal{L}(e^{-2t}\sin 3t) = \frac{-3}{2}\sin 3t(e^{-2t})$$

4) $\sinh kt \sin kt$

c). $t^2 \sin kt$

$U = t^2$ $\quad dV = \sin kt$

$dU = 2t\,dt$ $\quad V = \frac{-1}{k}\cos kt$

$$\left[-\frac{t^2}{k}\cos kt\right]_0^\infty + \int_0^\infty \frac{2t}{k}\cos kt\,dt$$

$U = \frac{2t}{k}$ $\quad dV = \cos kt\,dt$

$dU = \frac{2}{k}dt$ $\quad V = \frac{1}{k}\sin kt$

$$\left[-\frac{t^2}{k}\cos kt\right]_0^\infty + \left[\frac{2t}{k^2}\sin kt\right]_0^\infty - \frac{2}{k^2}\int_0^\infty \sin kt\,dt$$

$$+\left[\frac{2}{k^3}\cos kt\right]_0^\infty$$

$$\left[-\frac{t^2}{k}\cos kt\right]_0^\infty + \left[\frac{2t}{k^2}\sin kt\right]_0^\infty + \left[\frac{2}{k^3}\cos kt\right]_0^\infty$$

3. a. Use the Laplace transform method to solve the DE:

$y'' - 2y' = -4$, $y(0) = 0$, $y'(0) = 4$

$s^2\mathcal{L}\{Y\} - \cancel{sY(0)} - Y'(0) - 2s\mathcal{L}\{Y\} + \cancel{2Y(0)} = \frac{-4}{s}$

$\mathcal{L}\{Y\}(s^2 - 2s) = \frac{-4}{s} + 4$

5/10 $\mathcal{L}\{Y\} = 4\left(1 - \frac{4}{s}\right)/s(s-2)$

b. Given the DE $y'' + 2y' + 5y = e^{-t}\sin t$, $y(0) = 0$, $y'(0) = 1$, obtain an expression for $\bar{y} = \mathcal{L}\{y(t)\}$. DON'T SOLVE THE DE!

$s^2\mathcal{L}\{Y\} - \cancel{sY(0)} - Y'(0) + 2s\mathcal{L}\{Y\} - Y(0) + 5\mathcal{L}\{Y\}$

0 $\mathcal{L}\{Y\}(s^2 + 2s + 5) = e^{-t}\sin t$

$\mathcal{L}\{Y\} = \frac{e^{-t}\sin t}{\mathcal{L}^{-1}(s^2+2s+5)} = \frac{e^{-t}\sin t}{\mathcal{L}^{-1}(s^2+2s+5)}$ Laplace of both sides of eq.

$= \frac{e^{-t}\sin t}{\mathcal{L}^{-1}(s+1)^2+3}$

4 a. Show that $\frac{s}{(s^2+2as+a^2+b^2)} = \mathcal{L}\left\{\frac{1}{b}e^{-at}(b\cos bt - a\sin bt)\right\}$

$= \frac{s}{(s+a)^2+b^2}$

4/10 almost!

$\neq \mathcal{L}\{e^{-at}\}\left(\frac{b^2}{s^2+b^2} - \frac{as^2}{s^2+b^2}\right)$

$\neq \mathcal{L}\left\{\frac{1}{b}e^{-at}(b\cos bt - a\sin bt)\right\}$

b. If $f(s) = \mathcal{L}\{F(t)\}$, prove that $\mathcal{L}\{e^{at}F(t)\} = f(s-a)$

0

9/40

$\frac{42}{80}$ D+

Feb 17, 1970

BOB MARKS Box 365-2

1. Find the Laplace transform of the functions

$\frac{3}{4}$ a) [illegible] $\frac{\cancel{4}}{\cancel{4}+(s^2+2)}$

b) $\sin 4(t-1)\,\alpha(t-1)$ $\frac{4e^{-s}}{s^2+16}$ the same?

$f(t) = \sin 4t$

$\frac{7}{20}$ ~~c) $\sin 4(t-1)$~~ $\frac{4e^{-s}}{s^2+16}$

$f(t) = \sin 4t$

~~d) $\sin 4t\;\alpha(t-1)$~~

$f(t) = \sin 4t + 4$

e^{-sc}

~~e) [illegible]~~ $\mathcal{L}\{\,\} = \int_0^\infty$

2. Find the inverse Laplace transforms of

a) $\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$ $1 \equiv A(s^2+1) + (Bs+C)s$

$s = 0 \Rightarrow A = 1$

$s = i \Rightarrow 1 = (Bi+C)i$

$1 = -B \Rightarrow B = -1$

$C = 0$

$\frac{1}{s(s^2+1)} = \frac{1}{s} + \frac{s}{s^2+1}$

$\mathcal{L}^{-1}\{F(s)\} = 1 + \cos t$

~~b) $\frac{e^{-2s}}{s(s^2+1)}$~~ $= e^{-2s}\left[\frac{1}{s^2+1} + \frac{1}{s}\right] \Longrightarrow \cancel{\sin(t-2)}$ why?

~~c) $\frac{5s+2}{s^2(s-2)(s+1)}$~~

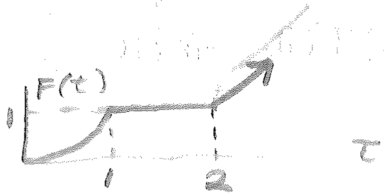
$\frac{6}{20}$

$\frac{1}{5}$ d) $\frac{4s}{(s-2)^2(s^2+4)}$ $\frac{\cancel{As+B}}{(s-2)^2} + \frac{Cs+D}{s^2+4}$

$4s = (As+B)(s^2+4) + (Cs+D)(s-2)^2$

$\frac{13}{40}$

in terms of unit step functions



$$F(t) = t^2 - t^2\alpha(t-1) + (t-1)\alpha(t-2) + 1\alpha(t-1) - 1\alpha(t-2)$$

19/20

$$\alpha(t-1)\left[1-t^2\right] + \alpha(t-2)\left[t-2\right] + t^2$$

$$f(t-1) = 1-t^2 \qquad f_2(t) = \cancel{0}\, t$$

$$f(t) = 1-(t+1)^2 \qquad \mathcal{L} = \frac{e^{-2s}}{s^2}$$

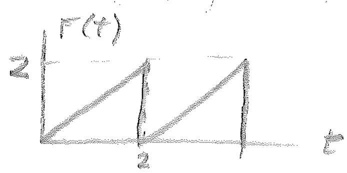
$$= -t^2 - 2t$$

$$\mathcal{L} = \left(-\frac{2}{s^3} - \frac{2}{s^2}\right)e^{-s}$$

$$\mathcal{L}\{F(t) = \frac{e^{-2s}}{s^2} - \left(\frac{2}{s^3} - \frac{2}{s^2}\right)e^{-s} + \frac{2}{s^3}$$

the periodic function $F(t)$ is defined by $F(t) = t$,

Find $\mathcal{L}\{F(t)\}$



$$F_1(t) = t - t\alpha(t-2)$$

$$g(t-2) = -t$$

$$g(t) = -(t+2)$$

$$\mathcal{L}\{g(t)\} = \frac{-1}{s^2} - \frac{2}{s}$$

10

$$\mathcal{L}\{F_1(t)\} = -e^{-2s}\left(\frac{1}{s^2} - \frac{2}{s}\right) + \frac{1}{s^2}$$

$$\mathcal{L}\{F(t)\} = \frac{-e^{-2s}}{1-e^{-2s}}\left(\frac{1}{s^2} - \frac{2}{s}\right) + \frac{1}{s^2(1-e^{-2s})}$$

0

29/40

$y(+0) = 1 \quad , \quad y'(+0) = 1$

$s^2\bar{Y} - sY(0) - Y'(0) + 4\bar{Y} = \frac{3s}{s^2+1}$

$s^2\bar{Y} - s - 1 + 4\bar{Y} = \frac{3s}{s^2+1}$

$\bar{Y}(s^2+4) = \frac{3s}{s^2+1} + s + 1$

$\bar{Y} = \frac{3s}{(s^2+4)(s^2+1)} + \frac{s}{s^2+4} + \frac{1}{s^2+4}$

$\frac{3s}{(s^2+4)(s^2+1)} = \frac{\cancel{3}s}{s^2+4} \quad \frac{-s}{s^2+1}$

$Y = 3\cos 2t + \cos t + \cos 2t + \frac{1}{4}\sin 2t$

$Y = 4\cos 2t + \cos t + \frac{1}{4}\sin 2t$

5/8

Solve the DE $y'' + 4y = F(t) \quad , \quad y(+0) = 0 \quad , \quad y'(+0) = 1$

$-1 + s^2\bar{Y} + 4\bar{Y} = \mathcal{L}\{F(t)\}$

$\bar{Y}(s^2+4) = \mathcal{L}\{F(t)\}$

$\bar{Y} = \frac{\mathcal{L}\{F(t)\}}{s^2+4}$

0

4) Solve the DE $y'' + 4y = 4t \quad , \quad y(+0) = 0 \quad y'(\pi) = 3$

$s^2\bar{Y} - B + 4\bar{Y} = \frac{4}{s^2}$

$\bar{Y}(s^2+4) = \frac{4}{s^2} + B$

$\bar{Y} = \frac{4}{(s^2+4)s^2} + \frac{B}{s^2+4}$

$= \frac{1}{s^2+4} - \frac{1}{s^2} + \frac{B}{s^2+4}$

$= \frac{B+1}{2}\sin 2t - t + \cancel{\frac{B}{2}\sin 2t}$

?/10

Bonus problem:

Solve $y'' + y = F(t) \quad , \quad y(+0) = 0 \quad , \quad y'(+0) = 0$ where $F(t) = \sec t$.

The solution is required in terms of elementary functions.

$s^2\bar{Y} + \bar{Y} = \mathcal{L}\{\frac{1}{\cos t}\}$

$\frac{41}{50}$ B

Name Bob Marks Box 385-2

1. Find the singular points of the D.E.

$$x^2(x-2)y'' + 3(x-2)y' + y = 0$$

and determine whether they are regular or irregular.

$X=0 \; ; X=2$

$$Y'' + \frac{3\cancel{(X-2)}Y'}{X^2\cancel{(X-2)}} + \frac{Y}{X^2(X-2)} = 0$$

$\leftarrow P(X) \qquad \leftarrow q(X)$

$X=0$ IS I.S.P. (POWER IN $P(X) \not\leq 1$)

$X=2$ IS R.S.P.

10

POWER OF $P(X) \leq 1$

POWER OF $q(X) \leq 2$

2. Solve the Euler equation

$$2x^2y'' + 3xy' - y = 0$$

$Y = X^S$

$2S(S-1)X^S + 3SX^S - X^S = 0$

$X^S(2S^2 - 2S + 3S - 1) = 0$

$\therefore 2S^2 + S - 1 = 0$

$2S^2 + S - 1$ $(2S+1)(S-1) = 0$

$S = -\frac{1}{2} \; ; S = -1$

$\frac{9}{10}$

$Y = a_0 Y^{+\frac{1}{2}} + a_1 Y^{-1}$

$\frac{19}{20}$

The [illegible] has a

regular singular point at $x=0$. Substitution of $y = \sum$ [illegible]

and rearrangement yields

$$(2c^2+c-1)a_0 x^c + \sum_{n=1}^{\infty}\left[(2n+2c-1)(n+c+1)a_n + 2(n+c)a_{n-1}\right]x^{n+c} = 0$$

(i) Find the indicial equation and determine its roots

(ii) For each root obtain the corresponding recurrence relation.

AT $n=0$ $\quad (2c^2+c-1)=0$

$(2c-1)(c+1)=0$ — IND. EQ.

$c_1=\frac{1}{2}\,;\; c_2=-1$

AT $n\geq 1$

$(2n+2c-1)(n+c+1)a_n = -2(n+c)a_{n-1}$

10

AT $c_2=-1$ $\qquad a_n = \dfrac{-2(n-1)a_{n-1}}{(2n-3)n}$

AT $c_2=\frac{1}{2}$ $\qquad a_n = \dfrac{-2(n+\frac{1}{2})a_{n-1}}{(2n)(n+\frac{3}{2})}$

4. The DE $y''-x^2y=0$ has an ordinary point at $x=0$. Substitute $y=\sum_{n=0}^{\infty} a_n x^n$ and obtain the recurrence relation between the coefficients a_n. Find the values of a_2 and a_3. What follows for the coefficients of the form a_{4k+2} and a_{4k+3}?

$$\sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$\sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=4}^{\infty} a_{n-4} x^{n-2} = 0$$

$a_3=0$ $\quad a_2=0$

$$a_3 6x + a_2 2 + \sum_{n=4}^{\infty} x^{n-2}\left[a_n n(n-1) - a_{n-4}\right] = 0$$

~~$6a_3 = -2a_2$~~

$$a_n = \frac{a_{n-4}}{n(n-1)}$$

5/10

(OVER)

15/20

$$a_n = \frac{a_{n-4}}{n(n-1)}$$

$$a_4 = \frac{a_0}{4 \cdot 3} \qquad a_5 = \frac{a_1}{5 \cdot 4}$$

$$a_8 = \frac{a_4}{8 \cdot 7} \qquad a_7 = \frac{a_3}{7 \cdot 6}$$

$$a_{12} = \frac{a_8}{12 \cdot 11} \qquad a_9 = \frac{a_5}{9 \cdot 8}$$

$$a_{4K} = \frac{a_0}{7 \cdot 56 \cdot 121 \cdots [(4n)(4n-1)]}$$

$$a_5 \cdot a_7 \cdot a_9 \cdots a_{4K+1} = \frac{a_1 \cdot a_3 \cdot a_5 \cdots a_{4K-1}}{(20 \cdot 42 \cdot 72 \cdots (4K+1)(4K)}$$

$$a_{4K+1} = \frac{a_1 \cdot a_3}{[20 \cdot 40 \cdot 72 \cdots (4K+1)(4K)}$$

5 The ODE [illegible] has an ordinary point at $x = 0$. Substitution of $y = \sum_{n=0} a_n x^n$ leads to the recurrence relation $a_n = \frac{2(n-4)(2n-5)}{n(n-1)} a_{n-2}$, a_0 and a_1 are arbitrary

(i) Show that for even values of n a terminating series results and obtain it

(ii) Show that for ODD values of n the resulting infinite series converges for $|x| < \frac{1}{2}$.

i) $a_2 = \frac{2(-2)(-1)}{2 \cdot 1}$

$a_4 = 0$

$\Rightarrow a_{2n} = 0 : n \geq 4$

5

ii) $a_3 = \frac{2(-1)(1)a_0}{3 \cdot 2}$

$a_5 = \frac{2(1)(5)a_3}{5 \cdot 4}$

$a_7 = \frac{2(3)(9)a_5}{7 \cdot 6}$

$n-4 \quad 2n-5$

$(2n+1)-4 \quad 2(2n+1)-5$

$2n-3 \quad 4n-3$

$\cancel{a_n = \frac{a_0(2^n)[-1 \cdot 1 \cdot 3 \ldots (2n-3)][1 \cdot 5 \cdot 9 \cdot (4n-3)]}{n!}}$

$\cancel{a_n = \frac{a_0 2^n [-1 \cdot 5 \cdot 27 \cdot \ldots \cdot (2n-3)(4n-3)]}{n!}}$

$(1-4x^2) = 0$

$\Rightarrow x = \pm\frac{1}{2}$

$\therefore$ EQ CONV. FOR $|x| < \frac{1}{2}$ why?

$\frac{2}{5} \quad \frac{7}{10}$

6 BONUS

If $a_n = \frac{2a_{n-1}}{n-1}$ determine a general expression for a_n in terms of a_0

$a_{n+1} = \frac{2a_n}{n} \Rightarrow \cancel{a_n = \frac{n a_{n+1}}{2}}$

$\cancel{a_0 = 0}$

$\cancel{a_1 = \frac{a_2}{2}}$

$\cancel{a_2 = \frac{2a_3}{2}}$

$\cancel{a_3 = \frac{3a_4}{2}}$

[illegible]

1. a) $\ldots = \frac{4}{(s+2)^2+16} = \frac{4}{s^2+4s+20}$

b) $\mathcal{L}\{\sin 4(t-1)\,\alpha(t-1)\} = \frac{4e^{-s}}{s^2+16}$

c) $\mathcal{L}\{\sin 4(t-1)\} = \mathcal{L}\{\sin 4t\cos 4 - \cos 4t\sin 4\} = \frac{4\cos 4 - s\sin 4}{s^2+16}$

d) $\mathcal{L}\{\sin 4t\,\alpha(t-1)\} = \mathcal{L}\{\sin[4(t-1)+4]\,\alpha(t-1)\}$

$= \mathcal{L}\{\cos 4\sin 4(t-1)\,\alpha(t-1) + \sin 4\cos 4(t-1)\,\alpha(t-1)\} = (4\cos 4 + s\sin 4)\frac{e^{-s}}{s^2+16}$

e) $\mathcal{L}\{\sqrt{t}\} = \mathcal{L}\{t\frac{1}{\sqrt{t}}\} = -\frac{d}{ds}\mathcal{L}\{\frac{1}{\sqrt{t}}\} = -\frac{d}{ds}\frac{\sqrt{\pi}}{\sqrt{s}} = \frac{1}{2}\frac{\sqrt{\pi}}{s\sqrt{s}}$

2. a) $\mathcal{L}^{-1}\{\frac{1}{s(s^2+1)}\} = \mathcal{L}^{-1}\{\frac{1}{s} - \frac{s}{s^2+1}\} = 1 - \cos t$

b) $\mathcal{L}^{-1}\{\frac{e^{-2s}}{s(s^2+1)}\} = [1-\cos(t-2)]\,\alpha(t-2)$

c) $\mathcal{L}^{-1}\{\frac{5s+2}{s^2(s-2)(s+1)}\} = \mathcal{L}^{-1}\{-\frac{2}{s} - \frac{1}{s^2} + \frac{1}{s-2} + \frac{1}{s+1}\} = -2 - t + e^{2t} + e^{-t}$

d) $\mathcal{L}^{-1}\{\frac{4s}{(s-2)^2(s^2+4)}\} = \mathcal{L}^{-1}\{\frac{1}{(s-2)^2} - \frac{1}{s^2+4}\} = te^{2t} - \frac{1}{2}\sin 2t$

3. a)

b) $F(t) = t^2[1-\alpha(t-1)] + [\alpha(t-1) - \alpha(t-2)] + (t-1)\alpha(t-2)$

$= t^2 - (t^2-1)\alpha(t-1) + (t-2)\alpha(t-2)$

$= t^2 - (t-1)^2\alpha(t-1) - 2(t-1)\alpha(t-1) + (t-2)\alpha(t-2)$

c) $f(s) = \frac{2}{s^3} - (\frac{2}{s^3} + \frac{2}{s^2})e^{-s} + \frac{1}{s^2}e^{-2s}$

[illegible]

$\mathcal{L}\{F_1(t)\} = \mathcal{L}\{t - [illegible]\} = \mathcal{L}\{t - (t-2)u(t-2) - 2u(t-2)\}$

$= \frac{1}{s^2} - \frac{1}{s^2}e^{-2s} - \frac{2}{s}e^{-2s}$

$\mathcal{L}\{F(t)\} = \frac{\mathcal{L}\{F_1(t)\}}{1-e^{-2s}} = \frac{1-e^{-2s}-2se^{-2s}}{s^2(1-e^{-2s})} = \frac{1}{s^2} - \frac{2e^{-2s}}{s(1-e^{-2s})}$

$= \frac{1}{s^2} - \frac{2e^{-s}}{s(e^{s}-e^{-s})} = \frac{1}{s^2} + \frac{1}{s} - \frac{1}{s}\coth s$

b) $t * e^t = \int_0^t (t-u)e^u\,du = (t-u)e^u\Big|_0^t + \int_0^t e^u\,du = (t-u)e^u\Big|_0^t + e^u\Big|_0^t$

$= (t-u+1)e^u\Big|_0^t = e^t - t - 1$

5 a) $s^2\bar{y} - s - 1 + 4\bar{y} = \frac{3s}{s^2+1} = (s^2+4)\bar{y} - s - 1$, $(s^2+4)\bar{y} = \frac{3s}{s^2+1} + s + 1$

$\bar{y} = \frac{3s}{(s^2+1)(s^2+4)} + \frac{s+1}{s^2+4} = \frac{s}{s^2+1} + \frac{1}{s^2+4}$

$y = \cos t + \frac{1}{2}\sin 2t$

b) $s^2\bar{y} - 1 + 4\bar{y} = \bar{F} = (s^2+4)\bar{y} - 1$, $(s^2+4)\bar{y} = \bar{F} + 1$

$\bar{y} = \frac{1}{s^2+4} + \frac{\bar{F}}{s^2+4}$

$y = \frac{1}{2}\sin 2t + \frac{1}{2}F * \sin 2t = \frac{1}{2}\sin 2t + \frac{1}{2}\int_0^t F(t-u)\sin 2u\,du$

c) Let $y'(+0) = A$ then

$s^2\bar{y} - A + 4\bar{y} = \frac{4}{s^2} = (s^2+4)\bar{y} - A$, $(s^2+4)\bar{y} = A + \frac{4}{s^2}$

$\bar{y} = \frac{A}{s^2+4} + \frac{4}{s^2(s^2+4)} = \frac{A-1}{s^2+4} + \frac{1}{s^2}$

$y = t + \frac{1}{2}(A-1)\sin 2t \qquad y' = 1 + (A-1)\cos 2t$

$y'(\pi) = 1 + (A-1) = A = 3$

$y = t + \sin 2t$

Pg 289)

1) $Y'' + Y = 0$

a) $s^2\bar{Y} - As - B + \bar{Y} = 0$

$\bar{Y}(s^2+1) = As - B$

$\bar{Y} = \frac{As}{s^2+1} - \frac{B}{s^2+1}$

$Y = A\cos X - B\sin X$

b) $Y = \sum_0^\infty a_n X^n$

$Y' = \sum_0^\infty a_n n X^{n-1}$

$Y'' = \sum_0^\infty n(n-1) a_n X^{n-2}$

$$\sum_0^\infty a_n X^n + \sum_0^\infty n(n-1) a_n X^{n-2} = 0$$

$$\sum_2^\infty a_{n-2} X^{n-2} + \sum_0^\infty n(n-1) a_n X^{n-2} = 0$$

$$\sum_{n=2}^\infty a_{n-2} X^{n-2} + \sum_{n=2}^\infty n(n-1) a_n X^{n-2} = 0$$

$$\sum_{n=2}^\infty [a_{n-2} + n(n-1)a_n] X^{n-2} = 0$$

$$\therefore\ a_{n-2} + n(n-1)a_n = 0 \Rightarrow a_n = \frac{-a_{n-2}}{n(n-1)}$$

a_0 | a_1

$a_2 = \frac{-a_0}{2\cdot 1} = \frac{-a_0}{2!}$ | $a_3 = \frac{-a_1}{3\cdot 2} = \frac{-a_1}{3!}$

$a_4 = \frac{-a_2}{4\cdot 3} = \frac{a_0}{4!}$ | $a_5 = \frac{-a_3}{5\cdot 4} = \frac{a_1}{5!}$

$a_6 = \frac{-a_4}{6\cdot 5} = \frac{-a_0}{6!}$ | $a_7 = \frac{-a_5}{7\cdot 6} = \frac{-a_1}{7!}$

$a_8 = \frac{-a_6}{8\cdot 7} = \frac{a_0}{8!}$ | $a_9 = \frac{-a_7}{9\cdot 8} = \frac{a_1}{9!}$

$$a_{2K} = \frac{(-1)^K a_0}{(2K)!}\ ;\quad a_{2K+1} = \frac{(-1)^K a_1}{(2K+1)!}\ ;\quad K \geq 1$$

$$Y = \sum_{K=1}^\infty a_{2K} X^{2K} + a_0 + \sum_{K=1}^\infty a_{2K+1} X^{2K+1} + a_1 X$$

$$= a_0\left[1 + \sum_{K=1}^\infty \frac{(-1)^K X^{2K}}{(2K)!}\right] + a_1\left[X + \sum_{K=1}^\infty \frac{(-1)^K X^{2K+1}}{(2K+1)!}\right]$$

$$= a_0 \sin X + a_1 \cos X$$

3) $Y'' + 3XY' + 3Y = 0$

$$Y = \sum_{n=0}^{\infty} a_n X^n$$

$$Y' = \sum_{n=0}^{\infty} n a_n X^{n-1}$$

$$Y'' = \sum_{n=0}^{\infty} n(n-1) a_n X^{n-2}$$

$$\sum_{n=0}^{\infty} n(n-1) a_n X^{n-2} + 3X \sum_{n=0}^{\infty} n a_n X^{n-1} + 3 \sum_{n=0}^{\infty} a_n X^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n X^{n-2} + 3X \sum_{n=1}^{\infty} n a_{n-1} X^{n-2} + 3 \sum_{n=2}^{\infty} a_{n-2} X^{n-2} = 0$$

$$\sum_{n=2}^{\infty} X^{n-2} \left[n(n-1) a_n + 3X n a_{n-1} + 3 a_{n-2} \right] = 0$$

$$a_n = \frac{-3X a_{n-1} - 3 a_{n-2}}{n(n-1)}$$

$a_0 \quad a_1$

$$a_2 = \frac{-3(X a_1 + a_0)}{2 \cdot 1} = \frac{-3X a_0}{2!} - \frac{3 a_1}{2!}$$

$$a_3 = -3(X$$

D I F F E R E N T I A L   E Q U A T I O N S   I I

F I N A L   E X A M I N A T I O N

March 18, 1970

NAME: MARKS, ROBERT   BOX NO. 385-2

INSTRUCTOR HOFFSOMMER

INSTRUCTIONS: Work all questions of Part I and two questions of Part II. Indicate which in Part II are to be graded.

DO NOT WRITE BELOW THIS LINE.

1	a	0	(10)
	b	–	(10)
2	a	9	(10)
	b	5	(10)
3	i	6	(6)
	ii	6	(6)
	iii	~~8~~ 6	(8)
4	i	6	(6)
	ii	5	(7)
	iii	5	(7)
5		20	(20)
6	a i	5	(5)
	ii	5	(5)
	b i	1	(5)
	ii	0	(5)
7	a i	3	(~~2~~ 3)
	ii	4	(~~5~~ 7)
	iii	2	(3)
	b i	2	(2)
	ii	4	(4)
	iii	4+	(4)

8	a i	–	(4)
	ii	–	(4)
	iii	–	(~~6~~ 8)
	iv	–	(4)
9		20	(20)
~~10~~		—	(20)
~~11~~	a i	—	(3)
	ii	—	(7)
	b i	—	(3)
	ii	—	(7)
12	a i	2	(2)
	ii	2	(2)
	iii	2	(2)
	iv	2	(2)
	v	2	(2)
	b i	2	(2)
	ii	1	(2)
	iii	2	(2)
	iv	2	(2)
	v	2	(2)

137+   C+

SHORT TABLE OF INTEGRALS

$$\int \cos kx \, dx = k^{-1} \sin kx \qquad k \neq 0$$

$$\int \sin kx \, dx = -k^{-1} \cos kx \qquad k \neq 0$$

$$\int x \cos kx \, dx = k^{-2} [\cos kx + kx \sin kx] \qquad k \neq 0$$

$$\int x \sin kx \, dx = k^{-2} [\sin kx - kx \cos kx] \qquad k \neq 0$$

$$\int x^2 \cos kx \, dx = k^{-3} [2kx \cos kx + (k^2x^2 - 2)\sin kx] \qquad k \neq 0$$

$$\int x^2 \sin kx \, dx = k^{-3} [2kx \sin kx - (k^2x^2 - 2)\cos kx] \qquad k \neq 0$$

$$\int x^3 \cos kx \, dx = k^{-4} [(3k^2x^2 - 6)\cos kx + (k^3x^3 - 6kx)\sin kx] \qquad k \neq 0$$

$$\int x^3 \sin kx \, dx = k^{-4} [(3k^2x^2 - 6)\sin kx - (k^3x^3 - 6kx)\cos kx] \qquad k \neq 0$$

$$\int e^{ax} \cos kx \, dx = \frac{e^{ax}}{a^2 + k^2} \cdot (a \cos kx + k \sin kx)$$

$$\int e^{ax} \sin kx \, dx = \frac{e^{ax}}{a^2 + k^2} (a \sin kx - k \cos kx)$$

$$\int \cos ax \cos bx \, dx = \frac{\sin(a+b)x}{2(a+b)} + \frac{\sin(a-b)x}{2(a-b)} \qquad a^2 \neq b^2$$

$$\int \sin ax \sin bx \, dx = -\frac{\sin(a+b)x}{2(a+b)} + \frac{\sin(a-b)x}{2(a-b)} \qquad a^2 \neq b^2$$

$$\int \sin ax \cos bx \, dx = -\frac{\cos(a+b)x}{2(a+b)} - \frac{\cos(a-b)x}{2(a-b)} \qquad a^2 \neq b^2$$

$$\int \cos^2 ax \, dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$$

$$\int \sin^2 ax \, dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$$

$$\int \sin ax \cos ax \, dx = -\frac{\cos 2ax}{4a}$$

$$\int e^{ax} \, dx = a^{-1} e^{ax}$$

$$\int xe^{ax} \, dx = a^{-2} (ax - 1)e^{ax}$$

$$\int x^2 e^{ax} \, dx = a^{-3} (a^2x^2 - 2ax + 2)e^{ax}$$

$$\int x^3 e^{ax} \, dx = a^{-4} (a^3x^3 - 3a^2x^2 + 6ax - 6)e^{ax}$$

$$\int x^4 e^{ax} \, dx = a^{-5} (a^4x^4 - 4a^3x^3 + 12a^2x^2 - 24ax + 24)e^{ax}$$

PART I <u>Work all 8 problems of this part.</u>

1. Find the interval of absolute convergence for the following power series. If the interval is finite, determine whether the series converges or diverges at the endpoints of the interval. Show the work leading to your answers.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}}$

$\displaystyle\lim_{n\to\infty}\left|\frac{(x-1)^n}{\sqrt{n}}\right| = 0$

not at all obvious and not even true for all x.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!}\left(\frac{x}{2}\right)^{2n+1}$

2. a) Determine the Taylor series expansion of $f(x) = \frac{1}{x}$ about the point $a = 2$, including the general term in $(x - 2)^n$

b) Find an expression for the remainder $R_n(x,2)$ for the above series.

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-2)^n}{n!} f^n(2)$$

$$f(x) = \frac{1}{a} \qquad f(2) = \frac{1}{2}$$

$$f'(x) = \frac{-1}{a^2} \qquad f'(2) = \frac{-1}{4}$$

$$f''(x) = \frac{2}{a^3} \qquad f''(2) = \frac{2}{8}$$

$$f'''(x) = \frac{-6}{a^4} \qquad f'''(2) = \frac{-6}{16}$$

$$f^{iv}(x) = \frac{+24}{a^5} \qquad f^{iv}(2) = \frac{24}{32}$$

$$\vdots \qquad \vdots$$

$$f^n(x) = \frac{n!(-1)^n}{a^n} \qquad f^n(2) = \frac{n!(-1)^n}{2^{n+1}}$$

$$\therefore f(x) = \sum_{n=0}^{\infty} \frac{(x-2)^n}{\cancel{n!}} \frac{(-1)^n \cancel{n!}}{2^{n+1}}$$

$$\Rightarrow \frac{1}{x} = \sum_{n=0}^{\infty} \frac{(x-2)^n(-1)^n}{2^{n+1}}$$

$$R(x,2) = \int_2^x \frac{(x-t)^n}{n!} f^{n+1}(t)\,dt$$

what is that ?

3. Find the Laplace transforms of

(i) $F(t) = t^3 + 3e^{2t} + \cos 2t$

$= 6\left(\frac{t^3}{6}\right) + 3e^{2t} + \cos 2t$

$\mathcal{L}\{F(t)\} = \frac{6}{s^4} + \frac{3}{s-2} + \frac{s}{s^2+4}$ ✓

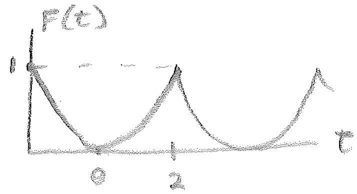
(ii) $F(t) = e^{-2t} \sin 3t$

$e^{at}(F(t)) \Rightarrow f(s-a)$

$\mathcal{L}\{\sin 3t\} = \frac{3}{s^2+9}$

$\mathcal{L}\{F(t)\} = \frac{3}{(s+2)^2+9} = \frac{3}{s^2+4s+13}$ ✓

(ii) $F(t) = (t - 1)^2$ for $0 < t < 2$, $F(t) = F(t + 2)$



$F^0(t) = (t-1)^2 \alpha(t-2)$

$g(t-2) = (t-1)^2$

$g(t) = (t+1)^2$

$= t^2 + 2t + 1$

$F_1(t) = (t-1)^2 [1 - \alpha(t-2)]$

$\mathcal{L}\{F'(t)\} = \left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right) e^{-2s}$

$\mathcal{L}\{F(t)\} = -\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right) \frac{e^{-2s}}{1-e^{-2s}}$

$+ \left(\frac{2}{s^3} - \frac{2}{s^2} + \frac{1}{s}\right) \frac{1}{1-e^{-2s}}$

4. Find the inverse Laplace transforms of

(i) $f(s) = \frac{1}{s^3} - \frac{1}{s-2} + \frac{1}{s^2+4}$

$\mathcal{L}^{-1}\{f(s)\} = \frac{t^2}{2} - e^{2t} + \frac{\sin 2t}{2}$ ✓

(ii) $f(s) = \frac{s^2 + 7s - 2}{s^3 + s^2 - 2s} = \frac{s^2+7s-2}{s(s^2+s-2)}$

$= \frac{s^2+7s-2}{s(s+2)(s-1)}$

$\frac{s^2+7s-2}{s(s+2)(s-1)} \equiv \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s-1}$

$s^2+7s-2 \equiv A(s+2)(s-1) + Bs(s-1) + Cs(s+2)$

$s=0 \Rightarrow {}^{-}2 = {}^{-}2A \Rightarrow A = 1$

$s=1 \Rightarrow 6 = 3C \Rightarrow C = 2$

$s={}^{-}2 \Rightarrow {}^{-}12 = 6B \Rightarrow B = {}^{-}2$

$\therefore f(s) = \frac{1}{s} - \frac{2}{s+2} + \frac{2}{s-1}$

$\mathcal{L}^{-1}\{f(s)\} = 1 - \cancel{\sqrt{2}\sin\sqrt{2}t} + 2e^{t}$

$2e^{-2t}$

$\frac{2}{s+2}$ not $\frac{2}{s^2+2}$

(iii) $f(s) = \frac{e^{-s}}{s(s+1)}$

$\cancel{f(s) = \frac{1}{a}}$ $\cancel{f(s-a) \Rightarrow e^{at}F(t)}$

$e^{-cs}f_*(s) = F(t-c)H(t-c)$

$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$

$1 = A(s+1) + Bs$

$s=0 \Rightarrow A = 1$

$s={}^{-}1 \Rightarrow B = {}^{-}1$

$\Rightarrow \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1} = f_*(s)$

$F(t) = 1 - \cancel{\sin t}\, e^{-t}$

$F(t-c)H(t-c) = (1 - \cancel{\sin(t-1)})H(t-1)$

$\therefore \mathcal{L}^{-1}\{f(s)\} = (1 - \sin(t-1))H(t-1)$

5. Use the Laplace transform method to solve the D.E.

$$y''(t) + y(t) = 3 \sin 2t \ , \quad y(+0) = 0 \ , \ y'(+0) = 1$$

$$s^2\bar{Y} - sY(0) - Y'(0) + \bar{Y} = \frac{6}{4+s^2}$$

$$s^2\bar{Y} - 1 + \bar{Y} = \frac{6}{4+s^2}$$

$$\bar{Y}(s^2+1) = \frac{6}{4+s^2} + 1$$

$$\bar{Y} = \frac{6}{(4+s^2)(s^2+1)} + \frac{1}{s^2+1}$$

$$\frac{6}{(4+s^2)(s^2+1)} = \frac{A}{4+s^2} + \frac{B}{s^2+1}$$

$$6 \equiv A(s^2+1) + B(s^2+4)$$

LET $s^2 = U$

$$6 \equiv A(U+1) + B(U+4)$$

$$U = {}^-1 \Rightarrow 6 = 3B \Rightarrow B = 2$$

$$U = {}^-4 \Rightarrow 6 = {}^-3A \Rightarrow A = {}^-2$$

$$\therefore \frac{6}{(4+s^2)(s^2+1)} = \frac{2}{s^2+1} - \frac{2}{s^2+4}$$

$$\Rightarrow \bar{Y} = \frac{3}{s^2+1} - \frac{2}{s^2+4}$$

$$Y = 3 \sin t - \sin 2t \quad \checkmark$$

6. a) The D.E. $(2 + x^2)y'' + xy' + 4y = 0$ has an ordinary point at $x = 0$

(i) Substitute $y = \sum_{n=0}^{\infty} a_n x^n$ and collect similar terms

(ii) Obtain the recurrence relation between the coefficients a_n

(You are NOT asked to determine the series solutions)

$2Y'' + X^2Y'' + XY' + 4Y = 0$

$$2\sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=0}^{\infty} a_n n(n-1)x^n + \sum_{n=0}^{\infty} a_n n x^n + 4\sum_{n=0}^{\infty} a_n x^n$$

$$\sum_{n=0}^{\infty} a_n x^n[n(n-1) + n + 4] + \sum_{n=0}^{\infty} 2a_n n(n-1)x^{n-2}$$

$$\sum_{n=2}^{\infty} a_{n-2}x^{n-2}(n^2-4n+8) + 2a_n n(n-1)x^{n-2}$$

$$\therefore a_{n-2}(n^2-4n+8) = 2a_n n(n-1)$$

$$a_n = \frac{a_{n-2}(n^2-4n+8)}{2n(n-1)} \quad ; n \geq 2$$

b) Substitution of the series $y = \sum_{n=0}^{\infty} a_n x^n$ in a certain D.E. gives the following results:

$$(n+2)(n+1)a_{n+2} - 4n(2n-1)a_{n-1} = 0 \quad \text{for } n \geq 1, \quad a_2 = 0$$

(i) Find the first two non-zero terms in the solution $y_1(x)$ for which $y_1(0) = 1$, $y_1'(0) = 0$ and also the first two non-zero terms in the solution $y_2(x)$ for which $y_2(0) = 0$, $y_2'(0) = 1$.

(ii) Find the interval of convergence for the series $y_1(x)$ and $y_2(x)$ by means of the ratio test, using the recurrence relation (investigation of the endpoints of the interval is not required).

$(n+2)(n+1)a_{n+2} = 4n(2n-1)a_{n-1}$

$$a_{n+2} = \frac{4n(2n-1)a_{n-1}}{(n+2)(n+1)}$$

7. a) Given the D.E. $4x^2y'' + 4xy' + (x^2 - 1)y = 0$

(i) Show that the origin is a regular singular point

(ii) Substitute $y = \sum_{n=0}^{\infty} a_n x^{n+c}$ and collect similar terms.

(iii) Find the indicial equation and the recurrence relation(s) for the coefficients a_n

(You are NOT asked to determine the series solutions)

i) SINGULAR PT = 0 AT X

$$y'' + \frac{y'}{x} + \frac{(x^2-1)y}{x^2} = 0$$

EXPONENT ≤ 1 EXPONENENT ≤ 2 ⇒ REG. SIN. PT.

ii) $\sum_0^\infty 4a_n(n+c)(n+c-1)x^{n+c} + \sum_0^\infty 4a_n(n+c)x^{n+c} + \sum_0^\infty a_n x^{n+c+2} - \sum_0^\infty a_n x^{n+c} = 0$

$\sum_0^\infty a_n x^{n+c}[(n+c)(4n+4c)-1] + \sum_0^\infty a_n x^{n+c+2}$

AT $n=0 \Rightarrow 4c^2 = 1 \Rightarrow c = \pm\frac{1}{2}$

$\sum_0^\infty x_n[(n\pm\frac{1}{2})(4n\pm 2)-1]a_n + \sum_2^\infty a_{n-2}x^{n+c}$

$a_1 = 0$

FOR $n \geq 2$: $a_n = \dfrac{-a_{n-2}}{1-(n\pm\frac{1}{2})(4n\pm 2)}$

OR $a_{n_1} = \dfrac{-a_{n-2}}{1-(2n+1)(2n+1)} = -a_{n-2}/[1-(2n+1)^2]$ AND $a_{n_2} = \dfrac{-a_{n-2}}{1-(2n-1)(2n-1)}$, $a_{n_2} = \dfrac{-a_{n-2}}{1-(2n-1)^2}$

b) The D.E. $2x(1-x)y'' + (1-x)y' + y = 0$ has a regular singular point at $x = 0$. Substitution of

$$y = \sum_{n=0}^{\infty} a_n x^{n+c}$$ and collecting similar terms gives

$$c(2c-1)a_0x^{c-1} + \sum_{n=1}^{\infty}(2n+2c-1)[(n+c)a_n - (n+c-2)a_{n-1}]x^{n+c-1} = 0$$

(i) Find the indicial equation and obtain its roots.

(ii) Find the recurrence relation between coefficients a_n for both roots of the indicial equation.

(iii) Show that one of the roots gives rise to a terminating series and obtain that series.

i) $c(2c-1) = 0$

$c_1 = \frac{1}{2}$ $c_2 = 0$

ii) FOR $c_2 = 0$

$(2n-1)[na_n - (n-2)a_{n-1}] = 0$ $n \geq 1$

$a_n = \dfrac{(n-2)a_{n-1}}{n}$

$a_1 = \dfrac{-1\ a_0}{1}$

$a_2 = 0$

$a_k = 0$ $k \geq 2$

$\therefore y = a_0x^0 - a_0x^1$

$= a_0 - a_0x$

FOR $c_1 = \frac{1}{2}$

$(n+\frac{1}{2})a_n = (\frac{1}{2}+n-2)a_{n-1}$

$(2n+1)a_n = (2n-3)a_{n-1}$

$a_n = \dfrac{(2n-3)a_{n-1}}{(2n+1)}$; $n \geq 1$

$a_1 = \dfrac{-1\ a_0}{3}$

$a_2 = \dfrac{1\ a_1}{5}$

$a_k = \dfrac{(-1\cdot 1\cdot 3\cdot 5\ldots(2n-3))a_0}{3\cdot 5\cdot 7\cdots(2n+1)} = \dfrac{-a_0}{(2n-1)(2n+1)}$

$y = a_0x^{\frac{1}{2}} - a_0\sum_{n=1}^{\infty}\dfrac{x^{n+\frac{1}{2}}}{(2n-1)(2n+1)}$

very good, but not required.

8. Given the function (A): $f(x) = \pi^2 - x^2$ for $-\pi \leq x \leq \pi$,

$f(x + 2\pi) = f(x)$

(i) Sketch the function in the interval $-3\pi < x < 3\pi$

(ii) Write a formula similar to (A) that represents the function in the interval $\pi \leq x \leq 3\pi$

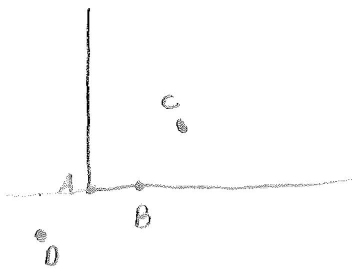
(iii) Expand $f(x)$ in a Fourier series.

(iv) By considering $f(0)$ show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{1}{12}\pi^2$$

PART II - Optional problems. Work any 2 problems of this part. If you work more than 2 and do not indicate which are to be graded, then the first 2 will be graded.

9. Find the best fitting straight line in the least squares sense to the points A(0,0) , B(1,0), C(2,2) and D(-1,-1). Plot the points and the line you found in a diagram.



A) $d_A = Y_A - mX_A - b = -b$

$(d_A)^2 = b^2$

B) $d_B = -(m+b)$

$(d_B)^2 = m^2+2mb+b^2$

C) $d_C = (2-2m-b)$

$(d_C)^2 = (2-2m-b)^2$

D) $d_D = -1+m-b$

$d_D^2 = (-1+m-b)^2$

$$f(m,b) = \sum(d^2) = b^2+(m+b)^2+(2-2m-b)^2+(-1+m-b)^2$$

$$0=\frac{\partial f}{\partial b} = 2b+2(m+b)-2(2-2m-b)-2(-1+m-b)$$

$$=2b+2m+2b-4+4m+2b+2-2m+2b$$

$$=8b+4m-2$$

$$0=\frac{\partial f}{\partial m} = 2(m+b)-4(2-2m-b)+2(-1+m-b)$$

$$=2m+2b-8+8m+4b-2+2m-2b$$

$$=12m+4b-10$$

$$24m+8b-20=0$$

$$4m+8b-2=0$$

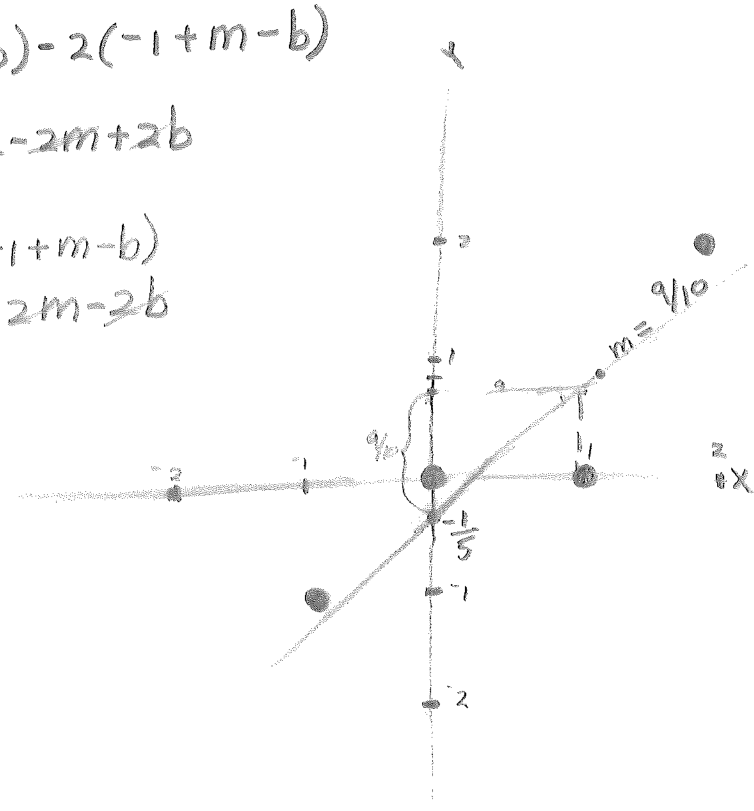
$$20m = 18 \Rightarrow m = 18/20 = 9/10$$

$$12m+24b=6$$

$$12m+4b=10$$

$$20b=-4$$

$$b = -4/20 = -1/5$$



BEST FIT: $Y=\frac{9}{10}X-\frac{1}{5}$ ✓

10. The Bessel function of order 1 is defined by the series

$$J_1(x) = \sum_{n=o}^{\infty} \frac{(-1)^n}{n!(n+1)!} \left(\frac{x}{2}\right)^{2n+1}$$

Compute $J_1(1)$ with an error less than .001

11. a. Given the function $F(t) = \begin{cases} 0 & , t < 0 \\ t & , 0 \leq t \leq 1 \\ e^{t-1} & , t \geq 1 \end{cases}$

(i) sketch the function

(ii) Express F(t) in terms of the unit step function.

b. Given the function

$$F(t) = t(2-t)\alpha(t) + (t-1)^2\alpha(t-1) + (\tfrac{1}{2}t - 1)\alpha(t-2)$$

(i) Sketch this function

(ii) Find its Laplace transform.

12. a) Determine whether the following functions are even, odd, or neither.

ODD $f(-x) = -f(x)$
EVEN $f(-x) = f(x)$

i) $(x + 1)^2$ NEITHER ✓

ii) x^3 - ODD ✓

iii) $\ln|\cos x|$ - EVEN ✓

iv) e^{-x^2} - EVEN ✓

v) $\sin(1 - x)$ - NEITHER ✓

i) $(-x+1)^2 \neq -(x+1)^2$
NOT ODD
$(-x+1)^2 \neq (x+1)^2$
NOT EVEN

ii) $(-x)^3 = -(x^3)$
ODD

iii) $\ln|\cos x| = \ln|-\cos x|$
EVEN

iv) $e^{-(-x)^2} = e^{-x^2}$
EVEN

v) $\sin(1+x) \neq -\sin(1-x)$
NOT ODD
$\sin(1+x) \neq \sin(1-x)$
NOT EVEN

b) Determine whether the following functions are periodic or not. If so, find their fundamental (lowest) period.

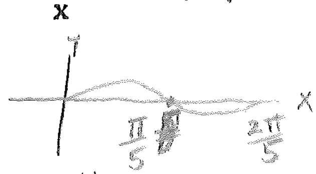
i) $\sin 5x$ - PERIODIC - $T = \frac{2\pi}{5}$ (ODD FUNCTION) ✓

ii) $\cos^2 x$ - PERIODIC - $T = \pi$ - (EVEN FUNCTION)

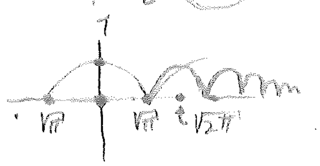
iii) $\cos x^2$ - NOT PERIODIC ✓

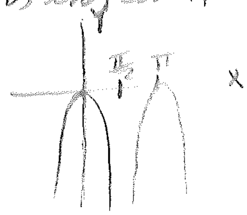
iv) $\ln|\cos x|$ → PERIODIC - $T = \pi$ (EVEN FUNCTION) ✓

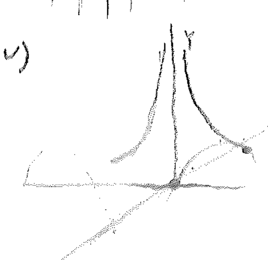
v) $\dfrac{\sin x}{x}$ → NOT PERIODIC ✓

i)


ii)


iii)


iv) $\ln|\cos x|$


v)


X

IN DETERMINATION OF ODD & EVEN PERIODIC FUNCTIONS, SINE & COSINE FORIER SERIES MAY BE USED RESPETIVELY, THUS ONLY ONE HALF OF THE GIVEN PERIOD NEED BE KNOWN.